Contractibility Design

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Abstract

We introduce a principal-agent model with costs of determining what is contractible. If there are front-end costs of distinguishing one action from another when writing contracts, then optimal contracts specify finitely many actions out of a continuum. This conclusion holds even when the cost of complete contracts is arbitrarily small but does not hold in the presence of arbitrarily large back-end costs of enforcing contracts. We apply our results to the design of employment contracts. Our model rationalizes the common practice of using discrete pay grades and predicts their rigidity in the face of small—but not large—productivity changes.

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1 Introduction

Contracts are not enforceable by *fiat*. Contractual enforceability obtains only under careful writing, ensuring the availability of evidence that can be used to prove or disprove the legality of a given action to an external arbitrator. Accordingly, the literature on contract law emphasizes the importance of *front-end* transaction costs of "foreseeing possible future contingencies, determining the efficient obligations that should be enforced in each contingency, [...] and drafting the contract language that communicates their intent to courts" (Scott and Triantis, 2005). The fact that lawyers spend up to 60% of their time drafting and reviewing documents (Thompson Reuters, 2024) underscores the importance of these front-end costs for contract design.

In practice, contracting parties weigh such burdens of complex contracts against their incentive gains. The outcome of this balancing act is that even billion-dollar commercial contracts are perplexingly vague, filled with phrases like "best efforts," "reasonable care," and "good faith" (Scott and Triantis, 2005). Yet, the textbook mechanism design approach to optimal incentive contracting (see *e.g.,* Bolton and Dewatripont, 2004; Laffont and Martimort, 2009) abstracts away from costs of writing contracts and does not speak to the fact that optimal contracts are so vague.

In this paper, we propose a framework for studying contractibility design in principal-agent settings. We model *contractibility* via a correspondence mapping what the principal asks the agent to do—or the "spirit" of the contract—to the set of actions that the agent can legally take following this request—or the "letter" of the contract. The principal designs contractibility to trade off the benefits of a finer-tuned contract against front-end costs.

We show that, under a large class of costs formalizing those incurred at the front-end, optimal contractibility is *coarse*, specifying only finitely many recommendations out of a continuum of possibilities. Optimal contracts are therefore vague: many actions are legally consistent with each of these finitely many recommendations. This conclusion holds even when the cost of perfect contractibility is arbitrarily small and the marginal cost of greater contractibility vanishes as contracts become complete. We show that it is the nature of costs—and not their magnitude—that generates contractual incompleteness.[1] More specifically, while front-end (fixed) costs generate incompleteness, back-end (variable) costs of enforcing contracts do not. We apply these results to study contractibility design for employment relationships. Our model rationalizes discrete pay grades, a common pay structure in practice (Bewley, 1999), as well as their rigidity in the face of productivity shocks.

[1] We adopt the definition of contractual incompleteness used, among others, by Spier (1992) and Scott and Triantis (2005). That is, while the contract specifies an obligation for each outcome, it *does not* specify a different obligation for each outcome, even when it is efficient to do so.

Model. A principal contracts with an agent who has private information. The principal writes a contract that specifies payments and payoff-irrelevant *recommendations*. The agent selects a recommendation and then takes the final, payoff-relevant action among those that are legal in light of the recommendation and the corresponding letter of the contract. The scope of contracts is specified by a *contractibility correspondence*, which describes all legal actions that the agent can choose after receiving a given recommendation.

The principal designs a contractibility correspondence in a pre-contractual stage. A contractibility correspondence is feasible only if it can be derived from an underlying evidentiary structure in which agents' actions generate evidence that the principal can use in court. The contractibility correspondence represents, for each initial recommendation, the set of actions that cannot be proven to be inconsistent with that recommendation. That is, the agent is innocent until proven guilty. The key economic assumption that defines the class of feasible contractibility correspondences is monotonicity: higher actions generate higher evidence.

Contractibility has a cost, reflecting the principal's efforts in writing the contract and building the evidentiary structure that makes it enforceable. As a leading example, we define a class of *costs of distinguishing actions* motivated by front-end costs: the principal pays a cost $g(x, y) > 0$ for every *possible* recommendation y and for every action x that they want to legally rule out under recommendation y. In the language of evidence, $g(x, y)$ is the cost the principal pays for the ability to produce some evidence under x that would be inconsistent with recommendation y.

The main assumption that we place on costs of contractibility is *strong monotonicity*. This property is most easily understood as regards the cost savings from ceasing to perfectly distinguish an interval of actions. In this case, strong monotonicity implies that the marginal cost of introducing perfect contracting for an interval of actions is (at most) second-order in the length of the interval. Every cost of distinguishing actions is strongly monotone. Intuitively, to distinguish t actions from t other actions, the number of costly comparisons that must be made is proportional to t^2.

Main Results. If costs are strongly monotone, then optimal contracts are coarse. In this case, the optimal contractibility correspondence specifies finitely many "grades," intervals of recommendations that allow the agent to take the same set of actions.

The intuition behind this result is that the benefits of contracting are *third-order* while costs are *second-order*. Specifically, to rule out intervals of perfect contractibility, we construct a payoff-improving alternative contractibility correspondence that introduces "local incompleteness," or replaces an interval with its two boundary points. For each type that was recommended an action in the interior of this interval, the principal was unconstrained by imperfect contractibility and therefore maximizing the virtual surplus function. Thus, there is no *first-order* loss in perturbing the assignment. To obtain the total loss in surplus, we integrate these second-order losses over the interval of types whose assignment changes, which is also proportional to the width of the interval—thus obtaining a *third-order loss*. This argument rules out intervals of perfect contractibility. More technical arguments based on similar set-valued perturbations rule out all other infinite sets.

Large contractibility costs are neither necessary nor sufficient for the conclusion that optimal contracts are coarse. The non-necessity of large costs follows from showing that strong monotonicity is consistent with costs satisfying both of the following properties: perfect contractibility is arbitrarily cheap and the asymptotic marginal cost of adding additional recommendations converges to zero. The non-sufficiency follows by showing that an alternative class of costs of contractibility, derived from *back-end* costs of contractual enforcement, yield complete contracts no matter their size. Concretely, we show that if the principal must pay $g(x, y)$ to distinguish x from y only in proportion to how likely it is that they would recommend y, then optimal contracts can be complete. This result highlights an important distinction between front-end (fixed) costs of contractibility and back-end (variable) costs of contractibility for optimal contractibility design.

We provide further results that describe how the economic primitives of the principal-agent problem affect both the coarseness of contracts and the design of the optimal contract. First, we derive an upper bound on the optimal number of contractible actions. This bound increases in the maximum concavity of the virtual surplus function because this scales the principal's loss from moving the agent's assignment and it decreases in the minimum complementarity of types with actions because this scales how tightly packed the principal's preferred allocations can be in small intervals. Second, we explicitly derive optimal coarse contracts using simple first-order conditions that equate the marginal benefits on virtual surplus of changing allocations with the marginal costs of refining contractibility.

Application: Employment Contracts. We use the model to study contractibility design in the workplace. Agents are workers who differ in their privately known productivity and can exert effort to produce output for a principal. The principal is a firm that uses incentive contracts to induce effort, but also bears front-end costs of designing them.

Our main result implies that the firm optimally employs "pay grades," discrete tiers of compensation that are common in practice, unlike piece rates (Bewley, 1999). We show that pay structures are *rigid* and unchanged in response to small changes to productivity, while large changes can induce a complete restructuring. Finally, the presence of incomplete information about worker productivity begets coarser contracts.

Related Literature. Our approach to modeling imperfect contractibility is inspired by the dichotomy between perfunctory performance (the letter of the contract) and consummate performance (the spirit of the contract) emphasized by Williamson (1975) and Hart and Moore (2008). We formalize this distinction via contractibility correspondences, which nest as a particular and special case the type of imperfect contractibility (free disposal) studied by Grubb (2009) and Corrao, Flynn, and Sastry (2023). Importantly, and differently from these papers, we study the *optimal* extent of contractibility. In addition, we micro-found the properties of contractibility from an evidentiary foundation that builds upon Green and Laffont (1986) and Hart, Kremer, and Perry (2017). Two new features of our evidentiary model are the endogeneity of evidence—evidence is endogenously generated by the agent's action, rather than exogenously by their type—and its optimal design.

Our work fits into a larger literature that provides foundations for incomplete contracts based on transaction costs (Simon, 1951; Coase, 1960). With respect to the classification of approaches described by Tirole (1999), our analysis shows how costs of writing contracts (*i.e.*, front-end costs) do lead to incompleteness while costs of enforcing contracts (*i.e.*, back-end costs) do not.[2]

Existing work on contracting with costly writing and/or enforcement studies problems with computational constraints on what events can be described (Anderlini and Felli, 1994, 1999; Al-Najjar, Anderlini, and Felli, 2006) and costs of writing that scale linearly with the number of clauses in a contract (Dye, 1985; Bajari and Tadelis, 2001; Battigalli and Maggi, 2002, 2008). Relative to this work, our analysis is different in two primary ways. First, we study an infinite (rather than finite) state and action space in which continuum contracts are feasible at finite cost. Thus, our analysis can directly speak to whether it is desirable to implement infinite contracts, like the extensively studied examples of piece rates for workers (Holmström, 1979; Holmström and Milgrom, 1987) and nonlinear pricing with smooth quantity discounts (Wilson, 1993). Second, as mentioned above, our analysis proposes a new framework for justifying costly contractibility based on costly evidence. This microfoundation for the form of contractibility costs is especially important in light of our result that some, but not all, costs lead to optimal coarseness.[3]

[2]There are, of course, other perspectives on why contracts may be incomplete. One such approach is based on the premise that parties can costlessly renegotiate a previously specified incomplete contract *ex post* (Segal, 1999; Hart and Moore, 1999; Che and Hausch, 1999). Another approach is based on the premise that *ex ante* costs of contracting serve a signaling role in the presence of private information for the principal (Spier, 1992). To avoid revealing information, the principal can optimally resort to incomplete contracts. This result is complementary to our analysis, where we show that—even in the absence of signaling effects—*ex ante* costs of contractibility generate optimally incomplete contracts.

[3]For example, Tirole (1999) raises the the possibility that the fixed cost per contingency proposed by Dye (1985) is *ad hoc* and restrictive because it rules out a continuum contract specifying constant wages (per unit of time) as infinitely costly.

Finally, our work is related to models in which a continuous variable is optimally discretized, such as in rational inattention (Jung, Kim, Matějka, and Sims, 2019), categorization (Mohlin, 2014), or simultaneous mechanism design and information design (Bergemann and Pesendorfer, 2007; Bergemann, Heumann, and Morris, 2022). A key step in our proof establishes novel bounds on the loss to the principal's payoff under their optimal mechanism from perturbations of contractibility. These bounds contribute to the literature on the loss from finite menus in nonlinear pricing models initiated by Wilson (1989) and generalized by Bergemann, Shen, Xu, and Yeh (2012), and Bergemann, Yeh, and Zhang (2021).

Outline. Section 2 introduces the model. Section 3 presents our main results. Section 4 applies our results to study optimal employment contracts. Section 5 presents a sketch of the proof of our main result on coarseness. Section 6 concludes.

2 Model

2.1 The Agent and the Principal

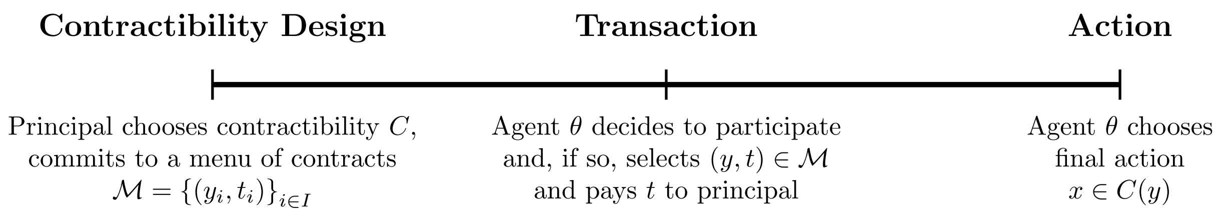
We build on a canonical principal-agent model. The agent's type $\theta \in \Theta = [0, 1]$ is drawn from a distribution F with strictly positive density f. The agent privately knows their type and can take an action x in the interval $X = [0, \overline{x}] \subset \mathbb{R}$. The agent has a twice continuously differentiable utility function $u : X \times \Theta \to \mathbb{R}$. We assume that higher types value higher actions more and that preferences are monotone increasing in the action: (i) u is strictly supermodular in (x, θ) and (ii) for each $\theta \in \Theta$, $u(\cdot, \theta)$ is strictly monotone increasing over X. The case with strictly decreasing preferences over X is analogous. All agent types value the zero action the same as their outside option payoff, which we normalize to zero, or $u(0, \theta) = 0$ for all $\theta \in \Theta$. The agent has quasilinear preferences over actions and money $t \in \mathbb{R}$, so their transfer-inclusive payoff is $u(x, \theta) - t$.

The principal's payoff derives from three sources. The first is the monetary payment $t \in \mathbb{R}$ from the agent. The second is a payoff that depends on the agent's action and type given by $\pi : X \times \Theta \to \mathbb{R}$, a twice continuously differentiable function such that $\pi(0, \theta) = 0$ for all $\theta \in \Theta$. The third is the cost of contractibility, which we will introduce in due course. We define the virtual surplus function $J : X \times \Theta \to \mathbb{R}$ as:

$$J(x, \theta) = \pi(x, \theta) + u(x, \theta) - \frac{1 - F(\theta)}{f(\theta)} u_\theta(x, \theta) \tag{1}$$

This is the total surplus generated when agent θ takes action x, net of the payments to the agent to ensure local incentive compatibility. We assume that J is twice continuously

Figure 1: Timeline of Events



differentiable, strictly supermodular in (x, θ), and strictly quasiconcave in x.[4] We assume supermodularity of the virtual surplus function in order to emphasize that the coarseness that will arise from contractibility design is an entirely distinct phenomenon from gaps in allocations that can occur under optimal bunching.

Example 1 (The Mussa and Rosen (1978) Linear-Quadratic-Uniform Model)**.** The payoffs and type distribution in Mussa and Rosen (1978) with $u(x, \theta) = x\theta$, $\pi(x, \theta) = -x^2/2$, and $F(\theta) = \theta$ satisfy all of our assumptions. △

2.2 Partial Contractibility

With perfect contractibility, a *contract* corresponds to a pair $(x, t) \in X \times \mathbb{R}$, composed of an enforceable action taken by the agent and the monetary transfer between the two parties. In our analysis, we relax the assumption that the principal can perfectly contract on actions.

To do so formally, we define a *contractibility correspondence* as a closed-valued and lower-hemicontinuous correspondence $C : X \rightrightarrows X$ that maps every recommendation $y \in X$ to a feasible set of final actions $x \in C(y)$ that the agent can take following that recommendation.[5] In our interpretation, C represents a codification of which actions can and cannot be proven to an external arbitrator as consistent with the request of the principal. More colloquially, y is the "spirit of the contract," which may differ from the full, legally allowable "letter of the contract" $C(y)$. With this, a contract is now a pair $(y, t) \in X \times \mathbb{R}$ composed of the initial, contractible recommendation and the monetary transfer. However, unlike the contractible recommendation, the action is only partially contractible and the agent can take any final action $x \in C(y)$.

The game between the agent and the principal is summarized in Figure 1. First, the principal chooses a contractability correspondence C and commits to a *menu* of contracts

[4] Sufficient primitive conditions that yield the maintained assumptions on J are that u and π are three times continuously differentiable with $u_{x\theta}, u_{xx\theta}, \pi_{x\theta} > 0$, $u_{xx}, u_{x\theta\theta}, \pi_{xx} < 0$, and that F has the increasing hazard rate property.

[5] Closed-valuedness and lower-hemicontinuity are technical conditions that will ensure the existence of an optimal contract.

$\mathcal{M} = \{(y_i, t_i)\}_{i \in I}$ for some arbitrary index set I. Second, the agent observes their private information $\theta \in \Theta$, decides whether to conduct a transaction with the principal and, if they do, picks a contract $(y, t) \in \mathcal{M}$. Third, the agent chooses a final action $x \in C(y)$. Finally, all payoffs are realized. The principal can equivalently choose a contractibility correspondence and commit to a *tariff* $T : X \to \overline{\mathbb{R}}$ assigning a transfer to every recommendation y, with prohibitively costly recommendations being the out-of-menu ones. We will often consider this equivalent formalization of the contract-design step.

Regular Contractibility Correspondences. We allow the principal to choose from a class of contractibility correspondences that discipline the relationship between the spirit and letter of a contract. To do this, we impose regularity conditions on contractibility that are motivated by a legal interpretation.

Definition 1 (Regularity)**.** *A contractibility correspondence C is regular if it satisfies:*

1. *Reflexivity: for every $y \in X$, $y \in C(y)$.*
2. *Excludability: $C(0) = \{0\}$.*
3. *Transitivity: for every $x, y, z \in X$, if $x \in C(y)$ and $z \in C(x)$, then $z \in C(y)$.*
4. *Monotonicity: for every $x, y \in X$, if $x \leq y$, then $C(x) \leq_{SSO} C(y)$.*[6]

We denote the set of regular contractibility correspondences by $\mathcal{C}$. *Reflexivity* requires that the agent *can* take action y when they are called upon to take action y by the contract. *Excludability* requires that the principal can always replicate the outside option by adding the contract $(0, 0)$ to the menu. *Transitivity* requires that, if an agent can reach action x by deviating from y and z by deviating from x, then they can reach z by deviating from y. *Monotonicity* requires that, if an agent is called upon to do $z \leq y$, then the set of things they can do at recommendation z is also lesser than the set of things they can do at y.[7] The following remark provides an evidentiary foundation for these conditions on contractibility:

Remark 1 (An Evidentiary Foundation for Regularity)**.** Regular contractibility correspondences arise as representing which actions the agent can take when the principal can use evidence generated by their actions to prove their contractual (in)consistency to an external arbitrator. Formally, an evidentiary correspondence $\mathcal{E} : X \rightrightarrows \Omega$ generates for every final

[6] $\leq_{SSO}$ denotes the strong set order.

[7] This is analogous to the standard monotone likelihood ratio property in moral hazard. Indeed, monotonicity of C is equivalent to the following discrete monotone likelihood ratio property defined with respect to set membership: For all $x, x', y, y' \in X$ such that $x \leq x'$ and $y \leq y'$,

$$\mathbb{I}[x' \in C(y')]\mathbb{I}[x \in C(y)] \geq \mathbb{I}[x' \in C(y)]\mathbb{I}[x \in C(y')]$$

action of the agent $x \in X$ a set of evidence $\mathcal{E}(x) \subseteq \Omega$. This evidentiary formalism follows closely that of Green and Laffont (1986) and Hart, Kremer, and Perry (2017), but differs in the crucial respect that evidence is endogenously generated by the action of the agent. The principal can prove that the action taken by the agent x was not consistent with the recommended action y if there exists a piece of evidence generated by the agent's action $\omega \in \mathcal{E}(x)$ that could not have been generated by the recommendation $\omega \notin \mathcal{E}(y)$. A court can sanction the agent and impose an arbitrarily large financial penalty if the principal can prove that the agent did not act in accordance with the recommendation. Internalizing this, if the agent is recommended to take action y, they would only ever take actions x that cannot be proven to be inconsistent with y. That is, the agent would only consider taking actions x such that $\mathcal{E}(x) \subseteq \mathcal{E}(y)$. Thus, the agent can take all (and only all) of the following actions without being proved to have deviated:

$$C_{\mathcal{E}}(y) = \{x \in X : \mathcal{E}(x) \subseteq \mathcal{E}(y)\} \tag{2}$$

An evidentiary correspondence then induces a contractibility correspondence as the set of actions to which the agent optimally restricts themselves in the so-called shadow of the law.

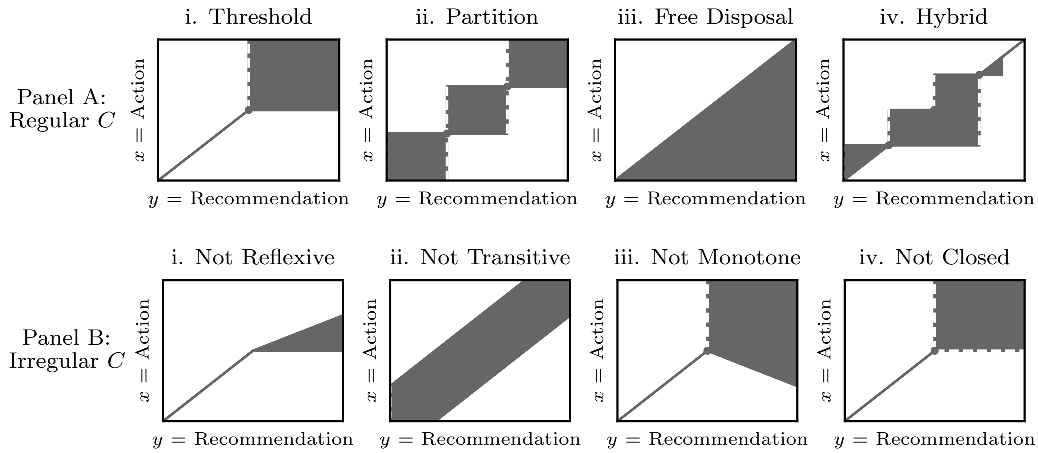
This structure immediately implies reflexivity and transitivity of $C_{\mathcal{E}}$. In Appendix C, we show that excludability and monotonicity are guaranteed whenever: (i) definitive evidence that the agent was excluded is available and (ii) higher actions by the agent generate higher pieces of evidence. This second assumption is an evidentiary analog of the standard condition in moral hazard models that higher actions are associated with higher distributions of signals. Conversely, we show that for any regular contractibility correspondence C, there exists an evidentiary correspondence $\mathcal{E}$ that satisfies (i) and (ii) such that $C = C_{\mathcal{E}}$. Thus, any regular C has an evidentiary foundation. △

Having argued that regularity of contractibility has a natural evidentiary foundation, we now illustrate what regularity rules out (and in) via a series of examples.

Example 2 (Illustrating Regularity)**.** We plot four examples of regular correspondences in Panel A of Figure 2. In the first regular example (i), all $x \leq 1/2$ can be specified perfectly in the contract, while all $x > 1/2$ are indistinguishable from one another. In example (ii), the action space is coarsened into four partitions of indistinguishable actions. In example (iii), the agent has access to unrestricted *free disposal* as studied by Grubb (2009) and Corrao, Flynn, and Sastry (2023). In example (iv), we combine these patterns into a "hybrid."

We also show four irregular examples in Panel B to better illustrate what our axioms rule out. Example (i) is not reflexive, since the correspondence does not include the 45 degree line; (ii) is not transitive, since there are "chains" whereby an agent can reach x from y and

Figure 2: Regular and Irregular Contractibility Correspondences



Note: Each graph illustrates a contractibility correspondence. Dark shading denotes the graph and dashed lines indicate open boundaries of the graph. The examples in Panel A (top row) are regular. The examples in Panel B (bottom row) are not regular.

z from x but not z from y; (iii) is not monotone, for $x > 1/2$; and (iv) is not closed, since $C(x)$ is open at the lower boundary for all $x > 1/2$. △

We now provide a mathematically convenient characterization of regular contractibility correspondences that clarifies their properties.

Proposition 1 (Representing Contractibility)**.** *A contractibility correspondence C is regular if and only if it can be written as $C(y) = [\underline{\delta}(y), \overline{\delta}(y)]$, where $\underline{\delta} : X \to X$ is an upper semi-continuous increasing function and $\overline{\delta} : X \to X$ is a lower semi-continuous increasing function such that: (i) $y \in [\underline{\delta}(y), \overline{\delta}(y)]$, (ii) $\underline{\delta}(x) = \underline{\delta}(y)$ for all $x \in [\underline{\delta}(y), y)$, (iii) $\overline{\delta}(x) = \overline{\delta}(y)$ for all $x \in (y, \overline{\delta}(y)]$, and (iv) $\overline{\delta}(0) = 0$.*

Proof. See Appendix A.1. □

This result characterizes regular contractibility correspondences in terms of their upper and lower envelopes, $\overline{\delta}(y) = \max C(y)$ and $\underline{\delta}(y) = \min C(y)$. The monotonicity of these functions is necessary by Monotonicity. Property (i) implies that $C(y)$ contains y, which is necessary by Reflexivity. Properties (ii) and (iii) come from Transitivity and are most easily understood via the graphical illustrations of Figure 2: if either of the envelopes deviate from the identity line, then they must be flat. In this sense, imperfect contractibility in our model always presents as "disposal" ("lower triangles"), "creation" ("upper triangles"), or complete indistinguishability ("boxes"). Finally, property (iv) ensures that $C(0) = \{0\}$, which is required by Excludability.

The properties of $\overline{\delta}$ and $\underline{\delta}$ imply that their images uniquely define these functions. More formally, given $\overline{D} = \overline{\delta}(X) \subseteq X$ and $\underline{D} = \underline{\delta}(X) \subseteq X$, one can recover uniquely recover the boundaries of C via the following operations: $\underline{\delta}(x) = \max_{z \leq x: z \in \underline{D}} z$ and $\overline{\delta}(x) = \min_{z \geq x: z \in \overline{D}} z$. Moreover, both $\underline{D}$ and $\overline{D}$ are closed sets and equal to the fixed points of $\underline{\delta}$ and $\overline{\delta}$, respectively.[8] Indeed, these sets correspond to the recommendations that an agent with monotone decreasing and increasing preferences over final actions, respectively, would follow. We therefore call these images the *sets of self-enforcing recommendations*.[9]

2.3 Costly Contractibility

As we have motivated, front-end costs are practically important for the design of contracts (Scott and Triantis, 2005). These costs of contractibility reflect the cost of introducing the legal and organizational apparatus that makes provable when actions are (not) consistent with the letter of a contract. We express this cost via the function $\Gamma : \mathcal{C} \to [0, \infty]$.

We can equivalently define each Γ over the space of pairs of functions $(\underline{\delta}, \overline{\delta})$ satisfying all the properties in Proposition 1. We endow this space with the relative topology induced by the L_1-norm over pairs of bounded and measurable functions over X.[10] For the rest of the analysis, we assume that the cost function Γ is lower semi-continuous in this topology. This means that two contractibility correspondences with finite cost whose maximum and minimum feasible deviations are close on average induce similar costs for the principal.

Our later analysis will show how additional conditions on the monotonicity and smoothness of Γ translate into properties of optimal contractibility. For now, we provide a few examples of continuous costs and discuss their economic interpretations.

Costs of Distinguishing Actions. Consider a principal who, for every possible spirit of the contract y, must differentiate the allowed actions within the letter of the contract, $C(y)$, from the disallowed actions outside of the letter of the contract, $X \setminus C(y)$. Specifically, suppose that it costs the principal some amount $g(x, y) > 0$ to distinguish any given recommendation y from every action that is not allowed under this recommendation $x \in X \setminus C(y)$,

[8] All these properties are formally established by Lemma 11 in Appendix B.

[9] The functions characterized in Proposition 1 are reminiscent of the properties of incentive compatible assignments in the literature on optimal delegation (see *e.g.,* Melumad and Shibano, 1991; Alonso and Matouschek, 2008). On the one hand, the properties of $\underline{\delta}$ and $\overline{\delta}$ are directly pinned down by the primitive properties of the contractibility correspondences rather than by incentive compatibility considerations. On the other hand, and somehow similarly to the delegation literature, these properties allow us to recast contractibility design as the design of the sets of self-enforcing recommendations.

[10] This is the norm defined by $||(\underline{\delta}, \overline{\delta})||_1 = \int_X \left(|\underline{\delta}(y)| + |\overline{\delta}(y)|\right) \mathrm{d}y$. The topology induced by this norm is the same as the product topology induced by endowing each space with the standard unidimensional L_1-norm.

where $g : X \times X \to \mathbb{R}_{++}$ is continuous.[11] We let $\mathcal{G}$ denote the set of such functions g. In the language of our evidentiary foundation (see Remark 1), this is equivalent to paying $g(x, y)$ whenever there exists a piece of evidence $\omega \in \mathcal{E}(x)$ such that $\omega \notin \mathcal{E}(y)$, *i.e.,* when the principal can distinguish x from y by using the generated evidence. Thus, these costs accord with the argument from Scott and Triantis (2005) that a designer may incur "high negotiation and drafting costs" to "partition all contingencies sufficiently" (p. 191). The cost of distinguishing outcomes is then the total cost of all such distinguishments:

Definition 2 (Costs of Distinguishing Actions)**.** *For any $g \in \mathcal{G}$, the cost of distinguishing outcomes is given by:*

$$\Gamma^g(C) = \int_X \int_{X \setminus C(y)} g(x, y) \, \mathrm{d}x \, \mathrm{d}y \tag{3}$$

Graphically, this is the area above and below the graph of C weighted by $g(x, y)$. Therefore, the cost of no contractibility and perfect contractibility are respectively equal to 0 and $\int_X \int_X g(x, y) \, \mathrm{d}x \, \mathrm{d}y$. Owing to Proposition 1, we can express this cost in terms of $(\overline{\delta}, \underline{\delta})$:

$$\Gamma^g(\underline{\delta}, \overline{\delta}) = \int_X G(\underline{\delta}(y), y) \, \mathrm{d}y + \int_X \left(G(\overline{x}, y) - G(\overline{\delta}(y), y) \right) \mathrm{d}y \tag{4}$$

where $G(x, y) = \int_0^x g(z, y) \, \mathrm{d}z$ and we observe Γ^g is continuous and *always finite*, that is, perfect contractibility is not, in principle, prohibitively costly to obtain. A natural special case is a constant cost of distinguishing actions.

Example 3 (Linear Costs)**.** Suppose that $g(x, y) = \kappa > 0$, *i.e.,* all pairs of actions are equally challenging to distinguish. In this case, the cost of distinguishing is simply proportional to the area of the graph above and below C, which is the area under $\underline{\delta}$ plus the area above $\overline{\delta}$:

$$\Gamma^\kappa(\underline{\delta}, \overline{\delta}) = \kappa \left(\int_X \underline{\delta}(y) \, \mathrm{d}y + \int_X \left(\overline{x} - \overline{\delta}(y) \right) \mathrm{d}y \right) \tag{5}$$

This cost function has the additional property of being linear in $(\underline{\delta}, \overline{\delta})$. For the regular contractibility correspondences in Panel A of Figure 2, the linear cost of distinguishing is proportional to the unshaded area (*i.e.*, the complement of the graph of C). △

Uncertain Costs of Distinguishing Actions. In the same setting as above, assume now that the principal must choose their legal and technological apparatus before knowing

[11] Observe that this implies that $g(x, x) > 0$. This has no practical implications. Indeed, by inspection of Equation 3, nothing in our analysis changes if we considered $\hat{g}(x, y) = g(x, y)\mathbb{I}[x \neq y]$, *i.e.,* ruling out any $x \neq y$ has strictly positive cost while $\hat{g}(x, x) = 0$.

exactly how costly it is to distinguish pairs of actions. We model this uncertainty of the principal through a prior distribution $\mu \in \Delta(\mathcal{G})$ with compact support.[12]

Definition 3 (Uncertain Costs of Distinguishing Actions)**.** *For any compactly supported $\mu \in \Delta(\mathcal{G})$ and $\lambda \in [-\infty, \infty]$, the uncertain cost of distinguishing outcomes is given by:*

$$\Gamma^{\mu,\lambda}(C) = \frac{1}{\lambda} \log \left(\int_{\mathcal{G}} \exp\left(\lambda \Gamma^{g}(C)\right) \mathrm{d}\mu\left(g\right) \right) \tag{6}$$

The interpretation is that the principal is a subjective expected utility maximizer in the face of uncertainty of the costs of distinguishing and their utility takes the CARA form. This nests the limit cases of $\lambda = 0$ and $\lambda = \infty$ that, respectively, correspond to the standard cost of distinguishing with the cost given by $g_\mu = \int_{\mathcal{G}} g \, \mathrm{d}\mu$ and to the Waldean worst-case scenario criterion. In general, $\Gamma^{\mu,\lambda}$ is continuous for any choice of (μ, λ).[13] Of course, risk-aversion ($\lambda > 0$) is the natural case, but we emphasize that it is neither concavity nor convexity that drive our results by allowing also for risk-loving preferences ($\lambda < 0$).

Foundations of Costs and Additional Cost Functions. In Appendix C, we show how to derive these cost functions and general properties of cost functions from an evidentiary foundation (as per Remark 1). Moreover, in Appendix D, we give several other classes of cost functions based on notions of costly enforcement, costly clauses, and menu costs.

2.4 The Principal's Problem

We now state the principal's mechanism and contractibility design problem. Given a fixed contractibility correspondence C, the revelation principle allows us to restrict to direct and truthful mechanisms.[14] Thus, a mechanism is a triple (ϕ, ξ, T) comprising a recommendation $\xi : \Theta \to X$, a final action or outcome $\phi : \Theta \to X$, and a tariff $T : X \to \overline{\mathbb{R}}$. The tariff and the recommendation jointly determine the transfer between the principal and the agent $T(\xi(\theta))$. The recommendation and the final action must be consistent with the contractibility correspondence, that is, $\phi(\theta) \in C(\xi(\theta))$. This, together with the usual incentive constraints, determines the set of implementable mechanisms.

Definition 4 (Implementable Mechanism)**.** *A mechanism (ϕ, ξ, T) is implementable given a contractibility correspondence C if the following three conditions are satisfied:*

[12] We endow the space $\mathcal{G}$ of strictly positive continuous real functions defined over $X \times X$ with the (relative) sup norm topology and we endow $\Delta(\mathcal{G})$ with the topology of weak convergence.

[13] This follows because the map $(\underline{\delta}, \overline{\delta}, g) \mapsto \Gamma^{g}(\underline{\delta}, \overline{\delta})$ is continuous and μ is supported on a compact set.

[14] Following the standard approach in mechanism design, we select the principal's preferred equilibrium and restrict to deterministic mechanisms. We omit the formal proof of the revelation principle in this case as it is standard and closely follows the steps in Myerson (1982).

1. *Obedience:*

$$\phi(\theta) \in \arg \max_{x \in C(\xi(\theta))} u(x, \theta) \quad \textit{for all} \;\; \theta \in \Theta \tag{O}$$

2. *Incentive Compatibility:*

$$\xi(\theta) \in \arg \max_{y \in X} \left\{ \max_{x \in C(y)} u(x, \theta) - T(y) \right\} \quad \textit{for all} \;\; \theta \in \Theta \tag{IC}$$

3. *Individual Rationality:*

$$u(\phi(\theta), \theta) - T(\xi(\theta)) \geq 0 \quad \textit{for all} \;\; \theta \in \Theta \tag{IR}$$

We let $\mathcal{I}(C)$ denote the set of implementable mechanisms given C.

Obedience requires that each agent θ chooses an optimal final action $\phi(\theta)$ by optimally exploiting what is possible under the contract given the initial recommendation $\xi(\theta)$, *i.e.,* they choose a favorite element from $C(\xi(\theta))$. Incentive Compatibility ensures that the agent wishes to actually select the recommendation $\xi(\theta)$ required by the mechanism, taking into account both the transfer they pay and their subsequent ability to optimize their final action within the scope described by the contract. Individual Rationality ensures that the agent is willing to participate in the mechanism.

Given C, the principal's problem is:

$$\mathcal{J}(C) := \sup_{(\phi, \xi, T) \in \mathcal{I}(C)} \int_{\Theta} \left(\pi(\phi(\theta), \theta) + T(\xi(\theta)) \right) \mathrm{d}F(\theta) \tag{7}$$

The principal's full problem balances the value and the cost of contractibility:

$$\sup_{C \in \mathcal{C}} \mathcal{J}(C) - \Gamma(C) \tag{8}$$

As this representation makes clear, designing contractibility and designing the contract are tightly linked, since contractibility determines what is implementable in the latter problem.

3 Optimal Contractibility and Optimal Contracts

In this section, we state our main results about optimal contractibility and contracts. First, we derive the value of any regular contractibility correspondence. Second, we introduce strong monotonicity of costs and show that all of the examples we have provided so far satisfy this condition. Third, we provide the main theorem on the existence of optimal contractibility and the necessity of its coarseness under strong monotonicity. Fourth, we characterize the allocation and transfer for an optimally coarse contract. Finally, we characterize which finite set of actions the principal chooses as contractible and describe an algorithm for computing the optimal contract and contractibility.

3.1 Optimal Contracts and the Value of Contractibility

We first study the problem of optimal contracting conditional on a given extent of contractibility. This also allows us to derive the value of any contractibility correspondence.

In principle, partial contractibility affects the problem in complex ways due to the interactions between Obedience and Incentive Compatibility, which allow for double deviations: when deciding what type to report, the agent takes into account their ability to later ignore the spirit of the contract (recommendation y) and instead take a different action within the letter of the contract ($x \in C(y), x \neq y$).

We first define the principal's favorite final action function $\phi^P : \Theta \to X$ as:

$$\phi^P(\theta) = \arg\max_{x \in X} J(x, \theta) \tag{9}$$

Moreover, for an arbitrary contractibility correspondence C represented by $(\underline{\delta}, \overline{\delta})$, we define the lowest implementable final action greater than $\phi^P(\theta)$ and the greatest implementable final action smaller than $\phi^P(\theta)$ as:

$$\overline{\phi}(\theta) = \min\{x \in \overline{D} : x \geq \phi^P(\theta)\} \quad \text{and} \quad \underline{\phi}(\theta) = \max\{x \in \overline{D} : x \leq \phi^P(\theta)\} \tag{10}$$

where we recall that $\overline{D} = \overline{\delta}(X)$ is the set of self-enforcing recommendations. With these objects in hand, we can now describe optimal contracts:

Proposition 2 (Optimal Contract)**.** *Fix a regular contractibility correspondence C with self-enforcing recommendations $\overline{D}$. Any optimal final action function is almost everywhere equal to:*

$$\phi^*(\theta) = \begin{cases} \overline{\phi}(\theta), & \text{if } J(\overline{\phi}(\theta), \theta) > J(\underline{\phi}(\theta), \theta), \\ \underline{\phi}(\theta), & \text{otherwise.} \end{cases} \tag{11}$$

Moreover, (ϕ^*, ϕ^*, T^*) is implementable with:

$$T^*(x) = u(x, (\phi^*)^{-1}(x)) - \int_0^{(\phi^*)^{-1}(x)} u_\theta(\phi^*(s), s)\, \mathrm{d}s \tag{12}$$

where $(\phi^*)^{-1}(z) = \inf\{\theta \in \Theta : \phi^*(\theta) \geq z\}$.

Proof. See Appendix A.2. □

We prove this result in three steps. In the first step, we show that a final action function ϕ is implementable if and only if it is monotone increasing in θ and its image satisfies $\phi(\Theta) \subseteq \overline{D}$. Intuitively, after being recommended any $y \in X$, the agent's favorite final action is $\overline{\delta}(y)$. Thus, if $y < \overline{\delta}(y)$, Obedience fails and the contract is not implementable. The substantive part of the proof establishes sufficiency by ruling out double deviations. The formula for the supporting tariff (Equation 12) follows from a standard application of the envelope theorem. In the second step, we combine our characterization of implementable final action functions with standard mechanism design arguments to reduce the principal's problem to an optimal control problem for the final action function. The third step characterizes the optimal final action function by solving this control problem. Intuitively, the optimal contract implements the "next best" thing to $\phi^P(\theta)$ that is actually contractible, in an incentive-compatible way. Our assumption that J is supermodular guarantees that this pointwise optimal policy is monotone and therefore globally optimal. As this result shows that ξ can be taken equal to ϕ, we henceforth focus on (ϕ, T) as the key objects of the contract.

Proposition 2 also shows that the value of any contractibility correspondence C depends only on its set of self-enforcing recommendations $\overline{D}$:

$$\mathcal{J}(C) = \int_\Theta J(\phi^*(\theta), \theta) \, \mathrm{d}F(\theta) = \int_\Theta \max_{x \in \overline{D}} J(x, \theta) \, \mathrm{d}F(\theta) \tag{13}$$

With some abuse of notation, we write $\mathcal{J} : \overline{\mathcal{D}} \to \mathbb{R}$ as the value of contractibility, where $\overline{\mathcal{D}}$ is the collection of sets of self-enforcing recommendations $\overline{D} = \overline{\delta}(X)$ induced by the upper envelopes of regular contractibility correspondences $C \in \mathcal{C}$. A consequence of Proposition 1 is that $\overline{\mathcal{D}}$ is equal to the set of *closed* subsets of X that contain both 0 and $\overline{x}$ (see Lemma 11 in Appendix B).

3.2 Monotone Costs of Contractibility

We now introduce our main assumption, *strong monotonicity*. This formalizes the idea that *marginal* costs of contractibility are always bounded away from zero. Consider first the case of cost functions defined over the real line. A cost function $\gamma : X \to \mathbb{R}$ is strongly monotone if there exists $\varepsilon > 0$ such that, for all $x, x' \in X$,

$$x' \geq x \implies \gamma(x') - \gamma(x) \geq \varepsilon(x' - x) \tag{14}$$

That is, the incremental cost is at least proportional to a linear function of the difference in the inputs. When γ is continuously differentiable, strong monotonicity is equivalent to having a derivative that is bounded away from zero: $\min_{x \in X} \gamma'(x) > 0$.

We extend this notion to our infinite-dimensional setting. First, we need to define linear functions of the "difference" in contractibility. Given two regular contractibility correspondences C' and C we write $C' \subseteq C$ if $C'(x) \subseteq C(x)$ for all $x \in X$, *i.e.,* C' exhibits *more contractibility* than C. In this case, $C \setminus C'$ is a well-defined correspondence from X to itself and will represent our notion of difference in contractibility. With this, for every C' and C such that $C' \subseteq C$, we define the linear function of the difference in contractibility as:

$$L(C \setminus C') = \ell(\text{Gr}(C \setminus C')) \tag{15}$$

where ℓ denotes the Lebesgue measure over $X \times X$ and $\text{Gr}(C \setminus C')$ denotes the graph of $C \setminus C'$. We argue that this is a natural notion of a linear function in our setting as L is linear in the difference between the graphs of the two correspondences, an object that fully describes the difference in contractibility between C' and C.[15]

Definition 5 (Strong Monotonicity)**.** *A cost function* Γ *is strongly monotone if there exists* $\varepsilon > 0$ *such that, for all* $C, C' \in \mathcal{C}$ *such that* $C' \subseteq C$*,*

$$\Gamma(C') - \Gamma(C) \geq \varepsilon L(C \setminus C') \tag{16}$$

Analogously to the real-valued case, in Appendix B.3 we show that if Γ is Gateaux differentiable in the appropriate sense, then it is strongly monotone when all its Gateaux derivatives are strongly monotone real functions (see Equation 14).

All of the examples of cost functions that we have given so far are strongly monotone:

Proposition 3. *Any (uncertain) cost of distinguishing actions is strongly monotone.*

Proof. See Appendix A.3 □

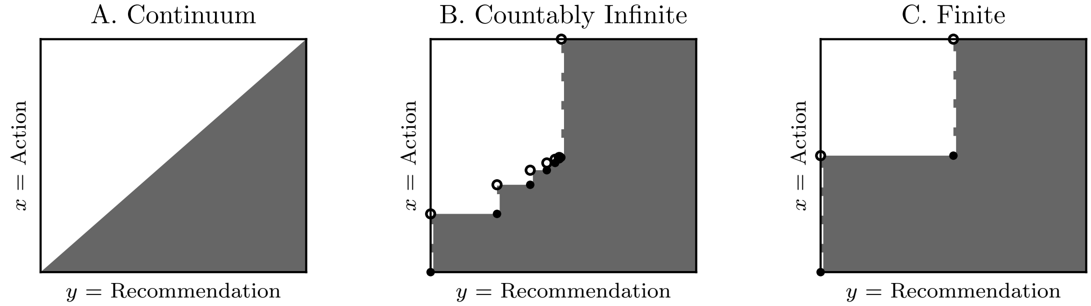
The key assumption on the cost of distinguishing actions that implies strong monotonicity is that the weighting function g is strictly positive. In particular, strong monotonicity fails only if the cost of distinguishing some pairs of outcomes is zero, *i.e.,* $g(x, y) = 0$ for a strictly positive Lebesgue measure set of $(x, y) \in X^2$, which is at odds with our premise that distinguishing outcomes is a costly activity.

3.3 Optimal Contractibility Exists and is Coarse

Given any $C \in \mathcal{C}$, we define its image as $C(X) = \{C(x)\}_{x \in X}$ and the cardinality of its image as $|C(X)|$. The cardinality function returns the simple count of the number of elements in

[15] Equivalently, this can be seen as L being linear in the difference between the boundaries of C' and C, *i.e.,* linear in $(\underline{\delta}' - \underline{\delta}, \overline{\delta} - \overline{\delta}')$.

Figure 3: Coarse and Non-Coarse Contractibility



A	B	C
$C(X) = \{[0,y]\}_{y\in[0,1]}$	$C(X) = \{[0, \frac{1}{2} - \frac{1}{2^k}]\}_{k=1}^{\infty} \cup \{[0,1]\}$	$C(X) = \{\{0\}, [0, \frac{1}{2}], [0,1]\}$
$\lvert C(X)\rvert = 2^{\aleph_0}$	$\lvert C(X)\rvert = \aleph_0$	$\lvert C(X)\rvert = 3$
Not Coarse	Not Coarse	Coarse

Note: Each panel shows an example contractibility correspondence. Panel A has the cardinality of the continuum, $2^{\aleph_0}$, and Panel B has the cardinality of the set of natural numbers, $\aleph_0$. These examples are not coarse and therefore incompatible with the conclusion of Theorem 1. Panel C has a finite cardinality. It is therefore coarse and compatible with the conclusion of Theorem 1.

$C(X)$ when it is finite, the cardinality of the natural numbers $\aleph_0$ when $C(X)$ is countably infinite, and the cardinality of the continuum $2^{\aleph_0}$ when $C(X)$ is uncountably infinite. Observe that $|C(X)|$ is a measure of the fineness (or coarseness) of the contractibility C. For example, every time that C exhibits a region of perfect contractibility, *i.e.*, $C(x) = \{x\}$ for all x in some open subset of X, then $C(X)$ is an uncountably infinite set. Conversely, when $|C(X)|$ is finite, only finitely many sets of actions can be distinguished in any contract. In other words, such a C corresponds to a legal and organizational apparatus that can prove a finite amount of statements only. We say that a contractibility correspondence C is *coarse* if $|C(X)|$ is finite. Figure 3 illustrates both non-coarse and coarse correspondences.

We now state our main result regarding optimal contractibility. To do this, we define the maximum concavity of virtual surplus $\bar{J}_{xx} = \max_{x,\theta} |J_{xx}(x,\theta)|$, the minimum complementarity of virtual surplus $\underline{J}_{x\theta} = \min_{x,\theta} J_{x\theta}(x,\theta)$, and the maximum density of types $\bar{f} = \max_\theta f(\theta)$. Finally, for every $r \in \mathbb{R}_+$, define $\lfloor r \rfloor = \max\{n \in \mathbb{Z} : n \leq r\}$.

Theorem 1 (Optimal Contractibility is Coarse)**.** *An optimal contractibility correspondence exists. If Γ is strongly monotone with constant $\varepsilon > 0$, then any optimal contractibility correspondence C^* is coarse:*

$$|C^*(X)| \leq 2 + \left\lfloor \frac{6\overline{x}\bar{J}_{xx}^2 \bar{f}}{\varepsilon \underline{J}_{x\theta}} \right\rfloor \tag{17}$$

An immediate implication of the theorem is the optimality of a coarse *contract*. In particular, by Obedience, we have that $\phi(\Theta) \subseteq \overline{\delta}^*(X)$ for any implementable—and therefore any optimal—final action function ϕ. Thus, any optimal final action function takes at most finitely many values and can be supported by a menu comprising an identical finite set of recommendations and an accompanying finite set of payments for each recommendation. Moreover, the proof of this result shows that it is optimal to set $\underline{D}^* = \{0\}$. Thus, it remains only to characterize the optimal choice of a finite set of self-enforcing recommendations $\overline{D}$ with a cardinality smaller than the bound provided by Theorem 1. We will discuss the properties of optimal and coarse contracts in Section 3.5.

Theorem 1 crystallizes the idea that the spirit and the letter of optimal contracts must differ. In fact, the optimally coarse contractibility correspondence associates each recommended action in the spirit of the contract with a set of legally admissible actions in the letter of the contract. This intentional imprecision is reminiscent of the vague language (*e.g.,* "best efforts," "reasonable care," and "good faith") that is "commonplace in commercial contracts" (Scott and Triantis, 2005, p. 196).

Main Idea of the Proof. We postpone the technical details of the proof of Theorem 1 until Section 5 and outline the key steps and logic here. Exploiting the relevant notions of continuity of the objective and compactness of the choice domain, we show that there exists an optimal contractibility correspondence. The main step in showing coarseness is establishing that an optimal set of self-enforcing recommendations cannot include any accumulation point (*i.e.*, limit point of an infinite sequence). If it did, we show by construction that there is a dominating, "coarsened" contractibility that removes all self-enforcing recommendations in a neighborhood of any accumulation point. The intuition behind this result is that the benefits of very precise contracting are an order of magnitude smaller than the costs. In particular, the increase in value from contracting in such a neighborhood (versus the coarsened construction that removes the interior of the neighborhood) is *third-order* in the size of the neighborhood, because the profit losses per agent are *second-order* and the measure of affected agents is *first-order*. The increase in costs from the same operation is *second-order* in the size of the neighborhood. In the context of costs of distinguishing, this is intuitive as removing a neighborhood of radius t eliminates the need to distinguish $2t$ actions from $2t$ actions, yielding a cost saving proportional to at least t^2. The same logic implies that for any coarse contractibility correspondence with $|\tilde{C}(X)| > 2 + \left\lfloor \frac{6\overline{x}\bar{J}_{xx}^2\bar{f}}{\varepsilon \underline{J}_{x\theta}} \right\rfloor$ we can construct an improving contractibility correspondence C that satisfies the bound in Equation 17.

3.4 Discussion of the Main Result

Before proceeding, we remark upon four notable points. The first two clarify the content of Theorem 1.

Remark 2 (Coarseness Obtains for Arbitrarily Small Costs)**.** Since ε in Definition 5 can be made arbitrarily small by scaling the cost function, Theorem 1 implies that optimal contractibility is coarse in the presence of arbitrarily small costs of contractibility. Formally, if Γ is strongly monotone, then $\kappa\Gamma$ is strongly monotone for all $\kappa > 0$. Naturally, if κ is small, then optimal contracts may specify a large, yet bounded, number of points. △

Remark 3 (Coarseness Obtains Despite the Potential "Concavity" of Costs)**.** Optimal coarseness would not be surprising if costs were assumed to be "convex" in some appropriate sense. However, as the following example shows, even when the marginal costs of additional contractibility converge to zero and costs are therefore "concave," cost functions can nevertheless be strongly monotone and optimal contractibility coarse.

Example 4 (Costs of Distinguishing Have a "Concavity" Property)**.** Suppose that the cost of contractibility is a linear cost of distinguishing and $\overline{x} = 1$. For any finite number of action recommendations $K \in \mathbb{N}$, we define C_K as the contractibility correspondence induced by the sets of self-enforcing recommendations $\underline{D} = \{0\}$ and $\overline{D} = \left\{\frac{k-1}{K-1}\right\}_{k=1}^{K}$, a uniform grid. A simple calculation demonstrates that:

$$\Gamma^{\kappa}(C_K) = \frac{\kappa}{2}\left(1 - \frac{1}{K-1}\right) \tag{18}$$

which is a *strictly concave* function of the number of contractible actions K. Indeed, as $K \to \infty$, $\Gamma^{\kappa}(C_K)$ approaches an asymptote of $\kappa/2$. △

Thus, even when the marginal cost saving of removing contractibility at infinity is zero, contracts can nevertheless be optimally coarse. △

We now make two remarks about when coarseness of contracts can fail to be optimal.

Remark 4 (Coarseness is About the Rate at Which Marginal Costs of Contractibility Converge to Zero)**.** The previous example shows that even when the asymptotic marginal cost of complete contracts is zero, it is nevertheless not optimal to design one. As our proof makes clear, this is because the economic benefits of more complete contracts are third-order while the costs of writing them are second-order under strong monotonicity. To formalize this intuition and clarify when coarseness can fail to obtain, consider the class of *clause-based costs* which depend on C only via its image's cardinality $|C(X)|$, *i.e.*, $\Gamma(C) = \hat{\Gamma}(|C(X)|)$

for some $\hat{\Gamma} : \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\} \to [0, \infty]$. Such costs are clearly not strongly monotone. Indeed, whether such costs generate coarseness depends on the returns-to-scale in contracting that they embody. With Proposition 13 in Appendix D.1, we formalize this by showing that if $\hat{\Gamma}(K) - \hat{\Gamma}(K-1)$ either does not converge to zero or converges to zero at a rate that is asymptotically less than order three, then contracts are optimally coarse. Conversely, Proposition 14 establishes that if $\hat{\Gamma}(K) - \hat{\Gamma}(K-1)$ converges to zero at a rate that is asymptotically greater than three, then contracts can be optimally complete. As a concrete example of this, consider the following:

Example 5 (Power Marginal Costs)**.** Suppose that the cost features increments that are some power of the number of clauses written, *i.e.,* $\hat{\Gamma}(K) - \hat{\Gamma}(K-1) = (K-2)^{\alpha}$ for some $\alpha \in \mathbb{R}$, which yields a cost $\hat{\Gamma}(K) = \sum_{k=1}^{K-2} k^{-\alpha}$.[16] This cost generates coarseness whenever $\alpha < 3$ and may otherwise fail to do so. △

This example highlights that with clause-based costs, which have been commonly applied in the incomplete contracts literature, the prediction of coarseness hinges on α, which represents returns-to-scale in contracting. For example, Battigalli and Maggi (2002) consider the cost $\hat{\Gamma}(K) = K$, which corresponds to $\alpha = 0$. By contrast, with costs motivated by front-end costs in legal contracting, such as costs of distinguishing, coarseness obtains regardless of assumptions on returns-to-scale. This is because such costs are guaranteed to be asymptotically second-order as long as distinguishing any pair of actions has a strictly positive cost. △

Remark 5 (Coarseness Obtains Because of the Front-End Nature of Costs)**.** We have so far argued that strong monotonicity is a weak property that is naturally satisfied by costs that have an interpretation as "front-end" costs of distinguishing what is allowed from what is not *ex ante*. We further justify this interpretation by showing that a natural and seemingly minor modification of costs of distinguishing to make these costs be borne *ex post* at the "back-end" renders optimal contractibility non-coarse. In contract law, Scott and Triantis (2005) identify "back-end" costs as the "expected cost of litigation" (p. 196), *i.e.,* the expected cost of proving that the agent did (or did not) take an action they were supposed to. To model this, for any recommendation function $\xi : \Theta \to X$, we define F_{ξ} as the induced distribution over recommendations in X. We can then define the back-end analog of a cost of distinguishing, which differs only in that the principal pays for each recommendation in proportion to how frequently it is assigned.

[16]To define this function for infinite cardinalities, we set $\hat{\Gamma}(2^{\aleph_0}) = \hat{\Gamma}(\aleph_0) = \sum_{k=1}^{\infty} k^{-\alpha}$.

Example 6 (Linear Back-End Costs)**.** As a concrete example, consider the linear *ex post* cost of distinguishing given by:

$$\Gamma(C, \xi) = \kappa \int_X \int_{X \setminus C(y)} \mathrm{d}x \, \mathrm{d}F_\xi(y) \tag{19}$$

for $\kappa > 0$. This is identical to the linear cost of distinguishing except $\mathrm{d}y$ is replaced by $\mathrm{d}F_\xi(y)$. This captures the interpretation that contracting costs are paid *ex post*, *i.e.,* after the agent is given a recommendation, and the principal considers the expected cost they will pay at the time of choosing C. △

Proposition 15 in Appendix D.2 shows that optimal contractibility is *not coarse* for such a back-end cost. This is despite the fact that for *any* fixed ξ with full range, $\Gamma(C, \xi)$ is strongly monotone. Intuitively, under an *ex post* cost, it is as if the principal incurs an additional profit loss from contracting, affecting optimal contracts but not contractibility. Thus, there is a qualitative distinction between front-end and back-end costs of contracting: frond-end (fixed) costs naturally yield coarse contracts, back-end (variable) costs do not. △

3.5 Designing Coarse Contracts

Having established that optimal contracts are necessarily coarse under strong monotonicity, we now study the problem of how a coarse contract can be optimally designed. Theorem 1 implies that any optimal contractibility correspondence can be represented via a coarse set of self-enforcing recommendations: $\overline{D} = \{x_1, \dots, x_K\}$ where $0 = x_1 < x_2 < \cdots < x_{K-1} < x_K = \overline{x}$. Define $\mathbf{x} = (x_1, \dots, x_K)$ and observe that the induced contractibility correspondence is $C(y) = [0, \overline{\delta}_{\mathbf{x}}(y)]$, where:

$$\overline{\delta}_{\mathbf{x}}(y) = \sum_{k=1}^{K} x_k \mathbb{I}[y \in (x_{k-1}, x_k]] \tag{20}$$

and we adopt, for expositional simplicity, the notational convention that $x_0 = 0$.

We first study how the optimal contract should be designed given a fixed $\overline{D}$. Applying Proposition 2 and using the specific structure of a coarse $\overline{D}$, we arrive at the following characterization of the optimal contract:

Proposition 4 (Coarse Contracts)**.** *Fix a coarse regular contractibility correspondence C with self-enforcing recommendations $\overline{D} = \{x_1, \dots, x_K\}$. Any optimal final action function is almost everywhere equal to:*

$$\phi^*(\theta) = \sum_{k=1}^{K} x_k \mathbb{I}[\theta \in (\hat{\theta}_k, \hat{\theta}_{k+1}]] \tag{21}$$

where for $k \in \{2, \ldots, K\}$, $\hat{\theta}_k$ is defined as the unique solution to $J(x_k, \hat{\theta}_k) = J(x_{k-1}, \hat{\theta}_k)$ if one exists, one if $J(x_k, \theta) < J(x_{k-1}, \theta)$ for all $\theta \in \Theta$, and zero if $J(x_k, \theta) > J(x_{k-1}, \theta)$ for all $\theta \in \Theta$, with the normalization that $\hat{\theta}_1 = 0$ and $\hat{\theta}_{K+1} = 1$.

Proof. See Appendix A.10. □

The K contractible actions are priced such that the types separate into a K-interval partition and the types in interval k purchase item k. The boundary types separating these intervals, $\{\hat{\theta}_k\}_{k=1}^K$, are such that the principal is indifferent between the agent's purchasing adjacent items, taking into account the marginal effect of that type's choices on the required information rents.

3.6 What Should Be Contractible?

We now study how the principal chooses which final actions are contractible. Momentarily, fix a finite set of self-enforcing recommendations $\overline{D} = \{x_k\}_{k=1}^K$ and its corresponding vector $\mathbf{x} = (x_k)_{k=1}^K$. As observed in Proposition 4, the optimal contract assigns action x_k to types $\theta \in (\hat{\theta}_k, \hat{\theta}_{k+1}]$. The principal's problem of choosing a contractibility correspondence with at most K self-enforcing recommendations is given by:

$$\max_{\{x_k\}_{k=1}^{\tilde{K}} : \tilde{K} \leq K} \mathcal{J}\left(\overline{\delta}_{\mathbf{x}}\right) - \Gamma(\overline{\delta}_{\mathbf{x}}) \tag{22}$$

where $\mathcal{J}\left(\overline{\delta}_{\mathbf{x}}\right) = \sum_{k=1}^K \int_{\hat{\theta}_k}^{\hat{\theta}_{k+1}} J(x_k, \theta) \,\mathrm{d}F(\theta)$ and $\Gamma(\overline{\delta}_{\mathbf{x}})$ are the principal's total value and cost, respectively.[17] To establish first-order conditions for optimal contractibility, we now supplement our assumptions on the cost of contractibility with differentiability:

Definition 6. *We say that Γ is finitely differentiable if, for every $K \in \mathbb{N}$, the map $\mathbf{x} \mapsto \Gamma(\overline{\delta}_{\mathbf{x}})$ is continuously differentiable at every vector $\mathbf{x} \in X^K$ such that $0 = x_1 < \ldots < x_K = \overline{x}$. In this case, we let $\frac{\partial}{\partial x_k}\Gamma(\overline{\delta}_{\mathbf{x}})$ denote the corresponding partial derivatives.*

Almost all of the examples of cost functions discussed in the previous sections are finitely differentiable.

Proposition 5. *Any uncertain cost of distinguishing actions is finitely differentiable provided that $\lambda \in (-\infty, \infty)$.*

Proof. See Appendix A.11. □

[17] Here we abuse notation by keeping the same symbol Γ to denote the section of the original cost function at $\underline{\delta} = 0$, that is, $\Gamma(\overline{\delta}_{\mathbf{x}}) = \Gamma(0, \overline{\delta}_{\mathbf{x}})$.

Finite differentiability allows us to derive a simple set of first-order conditions that optimal contractibility, hence optimal contracts, must satisfy:

Proposition 6 (Optimal Contractibility)**.** *If* Γ *is strongly monotone and finitely differentiable, then any optimal set of self-enforcing recommendations* $\overline{D}^* = \{x_1^*, \ldots, x_{K^*}^*\}$ *satisfies:*

$$\int_{\hat{\theta}_k}^{\hat{\theta}_{k+1}} J_x(x_k^*, \theta)\, \mathrm{d}F(\theta) = \frac{\partial}{\partial x_k}\Gamma(\overline{\delta}_{\mathbf{x}^*}) \qquad \textit{for } k \in \{2, \ldots, K^* - 1\} \tag{23}$$

Proof. See Appendix A.12. □

The left-hand-side of Equation 23 says that the marginal benefit of changing a self-enforcing recommendation x_k is the average increase in virtual surplus over all types allocated to that action. These marginal changes in virtual surplus take into account the direct effects on revenues and costs (holding fixed the agent's final action) as well as the indirect effects on the rest of the contract via information rents. A second effect of changing x_k, the change in the marginal types $\hat{\theta}_k$ and $\hat{\theta}_{k+1}$, is only second-order since the principal is indifferent between allocating those types either of two adjacent self-enforcing recommendations. The right-hand-side is simply the marginal cost of changing the self-enforcing recommendation x_k. Optimal contractibility balances these marginal benefits and costs.

Remark 6 (The Optimal Number Of Self-Enforcing Recommendations)**.** We have solved for optimal contractibility up to characterizing the optimal *number* of self-enforcing recommendations K^* and selecting among solutions to the first-order condition from Proposition 6 (if there are multiple). In this remark, we provide a practical method to find this number that we employ in our applications: (i) find the solutions that solve the first-order conditions for any fixed $K \leq B$, where B is the bound from Theorem 1; (ii) compute the best such solution for that K; and (iii) compare the values of the best solutions to find an optimal K^*.

First, for any $K \in \{2, \ldots, B\}$, define the set of candidate optima with K self-enforcing recommendations as those that solve the first-order conditions from Proposition 6:

$$\mathcal{O}_K := \left\{\mathbf{x} \in X^K : 0 = x_1 < \ldots < x_K = \overline{x} \text{ and Equation 23 holds}\right\} \tag{24}$$

Second, define the following value function as the value of the best candidate optimum with K self-enforcing recommendations:

$$\mathcal{V}(K) = \sup_{\mathbf{x} \in \mathcal{O}_K} \mathcal{J}\left(\overline{\delta}_{\mathbf{x}}\right) - \Gamma(\overline{\delta}_{\mathbf{x}}) \tag{25}$$

with the convention that $\mathcal{V}(K) = -\infty$ when $\mathcal{O}_K = \emptyset$. For $K = K^*$, this value coincides with the value of the original problem; moreover, we know that $K^* \leq B$. Third, we have that K^* solves the original problem if and only if

$$K^* \in \arg \max_{K \in \{2,\dots,B\}} \mathcal{V}(K) \tag{26}$$

which can be solved in linear time in the completeness bound, B. △

Thus, under strong monotonicity, we have reduced the question of optimal contractibility design from choosing a regular contractibility correspondence $C : X \rightrightarrows X$ to choosing a single number of self-enforcing recommendations $K \in \{2, \dots, B\}$ and we have established the structure of this problem along with a simple algorithm for its solution. In our applications below, we demonstrate the usefulness of this procedure in solving for calculate optimal contracts and contractibility under canonical assumptions.

4 Application to Employment Contracts

We now apply our results to study labor contracts with endogenous and costly contractibility. This is motivated by the fact that designing flexible pay structures requires front-end costs (Prendergast, 1999). We find that the firm optimally sets a coarse wage schedule that specifies a finite number of effort levels and corresponding payments, rather than fully flexible piece rates. This finding rationalizes discrete performance levels and pay grades, pay structures that are ubiquitous in practice (Bewley, 1999). We further find that the optimal pay structure can be rigid in the face of certain kinds of productivity shocks while changing discontinuously in the face of others. We show that incomplete information about worker productivity generates more coarse contracts. Finally, we describe additional applications to monopoly pricing and procurement.

4.1 Set-up: Labor Contracts with Imperfectly Contractible Effort

A worker (the agent) supplies labor to a firm (the principal). The worker's payoff from providing effort level $e \in E = [0, 1]$ is

$$\tilde{u}(e, \vartheta) = -a(1 - \vartheta)e - b\frac{e^2}{2} \tag{27}$$

where $\vartheta \sim \tilde{F} = U[0, 1]$ is the worker's privately observed productivity, $a > 0$ is a parameter that shifts productivity in a type-augmenting way, and $b > 0$ is a parameter that scales curvature in effort costs. High effort leads to more output for the firm, whose revenues are

$$\tilde{\pi}(e) = ce \tag{28}$$

where $c > 0$ represents the type-neutral productivity of effort. We introduce the simplifying assumption $b \leq c \leq 2a$ to ensure that the full action space is relevant for the problem: the principal will want to assign the highest effort to the highest productivity types and exclude the lowest productivity types from working at the firm. The firm faces front-end costs when writing the contract of the linear form introduced in Example 3:

$$\tilde{\Gamma}(\tilde{C}) = \kappa \int_E \int_{E \setminus \tilde{C}(\zeta)} \mathrm{d}e \, \mathrm{d}\zeta \tag{29}$$

where $\tilde{C} : E \rightrightarrows E$ is the contractibility correspondence and $\kappa > 0$. These are the costs of creating a legal and organizational structure that make different levels of effort enforceable.

4.2 Optimal Contracts Specify Discrete Performance Levels

We now apply our theoretical results to solve the firm's problem. As shown in Appendix A.13, we can apply the change of variables $x = 1 - e$ ("shirking") and $\theta = 1 - \vartheta$ ("unproductiveness") to bring the model under the assumptions of Section 2. That is, all workers prefer to shirk and unproductive workers prefer to do so to a greater extent. Since the linear cost of distinguishing outcomes is strongly monotone, Theorem 1 implies that any optimal contractibility correspondence is coarse. We can therefore optimize over a number $K \in \mathbb{N}$ of distinct shirking levels and a vector $\mathbf{x} = (x_k)_{k=1}^K$ specifying those levels.

We next use the structure of payoffs to more sharply describe the optimal contract. The principal's virtual surplus function is

$$J(x, \theta) = (2a\theta + b - c)x - b\frac{x^2}{2} \tag{30}$$

Restricted to coarse contracts, the costs of contractibility are

$$\Gamma(\overline{\delta}_{\mathbf{x}}) = \kappa \sum_{k=1}^{K-1} (1 - x_{k+1})(x_{k+1} - x_k) \tag{31}$$

Proposition 6 implies that optimal shirking levels solve a first-order condition, balancing the trade-off between the cost of specifying the contract *ex ante* and the benefits from screening *ex post*. Specifically, for each interior self-enforcing recommendation indexed by

$k \in \{2, K^* - 1\}$, we have that:

$$\int_{\hat{\theta}_k}^{\hat{\theta}_{k+1}} (2a\theta + b(1 - x_k) - c)\, \mathrm{d}\theta - \kappa(-2x_k + x_{k-1} + x_{k+1}) = 0 \tag{32}$$

where $\hat{\theta}_k = \frac{b}{4a}(x_k + x_{k-1}) + \frac{c-b}{2a}$ are the types between which the principal is indifferent to assign x_k or x_{k-1}. This equation reduces to a second-order difference equation that must be satisfied by the optimal shirking levels:

$$(x_{k+1} + x_{k-1} - 2x_k) \left[\frac{b^2}{16a}(x_{k+1} - x_{k-1}) - \kappa \right] = 0 \tag{33}$$

To solve for the optimal contract, we compute the solution of this difference equation (with boundary conditions $x_0 = 0$ and $x_K = 1$) for each candidate K. Using the algorithm described in Remark 6, we solve for the optimal K^*. This leads us to a closed-form characterization of the optimal contract:

Proposition 7 (Optimal Contracts Feature Discrete Pay Grades)**.** *The principal offers the following contract:*

$$e_k = \frac{k-1}{K^*-1} \qquad w(e_k) = \frac{1}{2}\frac{k-1}{K^*-1}\left(\frac{b}{2}\frac{k-1}{K^*-1} + c\right) \qquad k \in \{1, \ldots, K^*\} \tag{34}$$

where the optimal number of pay grades, K^*, satisfies $|K^* - \tilde{K}| < 1$ and

$$\tilde{K} = 1 + \frac{b^2}{12a\kappa} \tag{35}$$

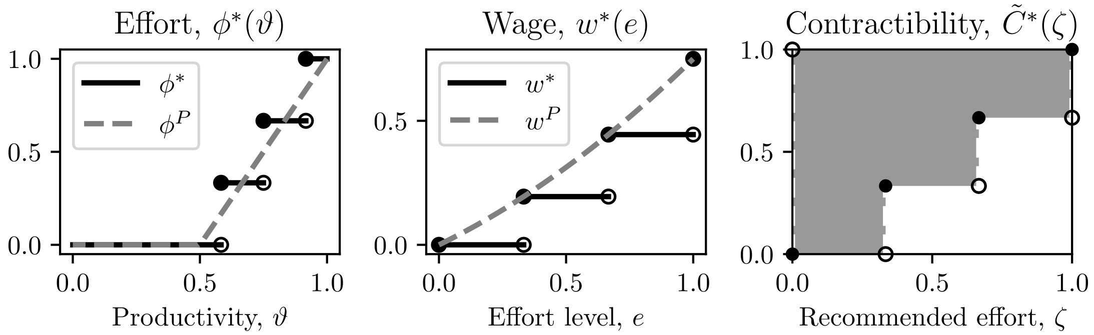
Moreover, K^* decreases in a, increases in b, decreases in κ, and does not depend on c. If $\frac{b^2}{16a\kappa} > 1$, then $K^* \geq 3$.

Proof. See Appendix A.13. □

Under the optimal contract, only a finite number of effort levels and wage payments are specified. We can interpret each contracted effort level as a "job" within the firm, associated with a distinct "pay grade." For each job, the spirit of the contract specifies e_k and the letter of the contract constrains the agent to exert equal or higher effort ($\tilde{C}^*(e_k) = [e_k, 1]$). In Figure 4, we illustrate the optimal assignment of effort and wages as well as the optimal contractibility. We compare this to the contract under perfect contractibility, which specifies a smooth relationship between continuously-specified effort levels and wage.

The optimality of uniformly spaced effort levels arises due to symmetries in the benefits and costs of more precise contracting. To understand the first property (symmetric benefits),

Figure 4: Optimal Wage Contracts with Costly Contractibility



Note: This figure illustrates the optimal contract from Proposition 7 with $a = b = c = 1$ and $\kappa = 1/32$. The contract is finite with $K^* = 4$. The first panel shows the assignment ϕ^*, in terms of the original problem of effort allocation, the second panel shows the wage w^*, and the third shows the contractibility correspondence $\tilde{C}^*$. In the first two panels, we also show the contract under perfect contractibility (ϕ^P, w^P).

we observe that the second derivative $J_{xx} = -b$ is constant as a function of (x, θ) and that the principal's optimal assignment absent contracting frictions induces a uniform distribution over actions. Starting from perfect contractibility, the opportunity cost of removing perfect contractibility in some interval of the action space is the same *regardless* of where that interval is located. This is for two reasons. First, as virtual surplus is quadratic in this model, the employer has an equal opportunity cost of forgoing differentiation for high-output versus low-output workers. Second, because the optimal assignment function is linear, the same measure of types is affected. The corresponding symmetry in costs arises because, for linear costs of distinguishing, producing evidence to distinguish high and low effort levels costs the same.

Interpretation: Pay Grades and Performance Bands. Our findings can help rationalize the ubiquity of pay grades that coarsely group workers to have common salaries. In his survey of wage-setting practices for US manufacturing and services firms, Bewley (1999) observes that piece rates that continuously vary compensation with output are relatively rare. A commonly mentioned problem with piece rates is the "cost of establishing the rates." Instead, an alternative arrangement is a "grade and step system" whereby the full set of labor tasks is segmented into discrete grades (job titles) and, within each grade, discrete steps that correspond to different salaries. Our model rationalizes such a system as an optimal response to even very small front-end costs of "establishing the rates." This contrasts with the prediction of the prediction of the model under perfect contractibility, a piece rate under which total wages vary smoothly with effort.

A specific example of a coarse compensation scheme are the performance bands or, more colloquially, "bonus buckets" used for analysts at investment banks. In these systems, employees are assigned a coarse performance grade (*e.g.,* "meets expectations" versus "exceeds expectations") at the end of a pay period and assigned a fixed bonus salary that corresponds to their grade. Through the lens of our model, performance grades may be an optimal compensation scheme in the presence of even small costs of distinguishing pieces of evidence regarding analysts' performance. These costs may be especially natural if an agent's contribution toward a goal (*e.g.,* executing a merger) is difficult to substantiate with hard, legally admissible evidence.

Our result also implies that coarse compensation can emerge even when revenues net of wage payments are arbitrarily large relative to front-end costs. This can be observed sharply from the invariance of coarseness K^* to the parameter c that scales revenue per unit of effort. This helps justify why coarse compensation schemes might persist even at very profitable enterprises, consistent with Bewley's (1999) observations for firms of variable size and from the ubiquity of vague performance standards in even high-value contracts.

4.3 Comparative Statics: Rigidity in Pay Structures and Wages

We now study how the optimal pay structure $\{e_k, w(e_k)\}_{k=1}^{K^*}$ responds to changes in type-augmenting productivity. To do this, we assume that parameters (a, b, c, κ) are such that there is a unique K^*. This is true for almost all such vectors of parameters.[18] Decreases in the parameter a increase the returns to effort for all agents, and do so by more for higher-productivity agents (*i.e.,* those with higher ϑ). The sizes of productivity shocks have markedly different implications for changes in the structure of pay:

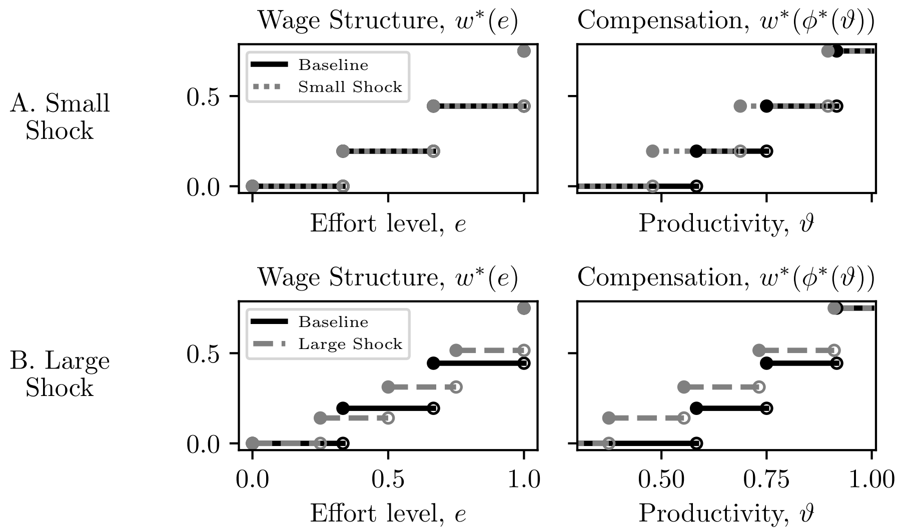
Corollary 1. *For every a, there exists a neighborhood $\mathcal{A}$ of a such that the optimal pay structure is invariant to a' for $a' \in \mathcal{A}$ and there exists a different optimal pay structure for $a' \notin \mathcal{A}$.*

Proof. See Appendix A.14 □

In response to shocks, both the number of wage levels and the wages within those levels are unaffected by small enough changes. When type-augmenting productivity increases by sufficiently small amounts, agents' wages may increase, but only through discrete jumps ("promotions") across fixed steps ("jobs"). We illustrate such a scenario in the first row

[18] This follows because if there are multiple optimal values for K^*, then they must differ by one. Given this, it is simple to show that, for a situation in which K^* is not unique, an indifference equation of the following form must hold: $\frac{b^2}{24a\kappa}\Lambda(K^*) = 1$, where Λ is a polynomial. As we consider changes in a, we maintain the assumption that $b \leq c \leq 2a$.

Figure 5: How Wage Structures Respond to Productivity Changes



Note: This figure illustrates the comparative statics of Corollary 1. We plot the wage structure ($w^*(e)$) and the compensation for each agent ($w^*(\phi^*(\vartheta))$) for a baseline case ($a = 1.0, K^* = 4$; black solid line), a "small shock" with slightly higher productivity ($a = 0.8, K^* = 4$; gray dotted line), and a "large shock" with much higher productivity ($a = 0.7, K^* = 5$; gray dashed line). We fix $b = c = 1$ and $\kappa = 1/32$. The graph of transfers is truncated at $\vartheta = 0.3$, as lower types are excluded and receive no wages in all cases.

(Panel A) of Figure 5. In response to larger changes, the entire pay structure can change: that is, the firm offers a different and potentially non-nested set of effort levels and corresponding wages. That is, in response to large increases in type-augmenting productivity, "jobs" are destroyed and created, and some individual workers may even be re-assigned to lower -wage "jobs." We illustrate such a scenario in the second row (Panel B) of Figure 5: in particular, in the right panel, note that the new compensation schedule (gray dashed line) is sometimes above and sometimes below the old schedule (black solid line). In this way, the model generates discrete transitions in the structure of pay in the presence of a continuous cost of determining the pay structure.

4.4 Incomplete Information Begets (More) Coarse Contracts

We next explore the interaction between incomplete information and incomplete contracts in our setting. We do this by comparing the optimal pay structure under adverse selection with the one under complete information. That is, the firm can perfectly determine the productivity of their workers and propose an allocation that depends on their actual productivity.[19] However, the firm must use the same extent of contractibility for all worker types, for example because the choice of contractibility must be made before the monopolist learns the ability of their employee(s).[20]

Under perfect contractibility, the monopolist would implement the efficient outcome that maximizes expected total surplus $S = \pi + u$ and compensate the worker exactly what is required to ensure individual rationality. Under costly contractibility, however, the principal may prefer to *imperfectly* differentiate to economize on the costs of writing a complex contract. We find that the efficient allocation also features uniform tiers and that there are more tiers than in the optimal allocation under incomplete information:

Corollary 2. *In the efficient contract, the optimal set of effort levels is* $\left\{\frac{k-1}{K^{*C}-1}\right\}_{k=1}^{K^{*C}}$ *where* $K^{*C} \geq K^*$*. Moreover,* K^{*C} *satisfies* $|K^{*C} - \tilde{K}^C| < 1$*, where* $\tilde{K}^C = 2\tilde{K} - 1$*.*

Proof. See Appendix A.15. □

This result implies that adverse selection results in both under-production and under-differentiation of performance levels relative to the efficient setting. This arises in our environment because more incomplete information dulls the firm's incentives to discriminate between types of workers, which in turn dulls the employer's incentive to contractually differentiate different effort levels. In Appendix B.4, we show that this logic is more general by providing conditions under which our coarseness bound from Theorem 1 is smaller under incomplete information than under complete information.

4.5 Additional Applications

The screening problem solved in this section admits other interpretations. In these settings, our model makes additional realistic predictions.

Procurement and Supply Chains. We can re-interpret our model such that the principal is a purchasing firm, the agent is a supplier whose costs are given by Equation 27, and the costly action is to produce an input of a given quality. Our result implies that a supplier contract specifies a coarse menu of quality levels and corresponding payments. Our result is consistent with Asanuma's (1989) description of input sourcing by automobile and machinery manufacturers in Japan. Asanuma (1989) describes how purchasing firms segment suppliers into three categories differentiated by the quality of their inputs and contract in

[19]In other words, we consider the complete-information setting where the feasible direct mechanisms satisfy Obedience and Individual Rationality, but not necessarily Incentive Compatibility.

[20]As we show in Appendix B.4, there is an alternative interpretation in which the worker rather than the firm has bargaining power (*i.e.*, monopoly rather than monopsony).

sharply different ways with suppliers in different categories. Through the lens of our model, this coarsening can be understood as an optimal reaction to even small costs of contractually distinguishing inputs of different qualities.[21]

Monopoly Pricing. As observed in Example 1, the Mussa and Rosen (1978) nonlinear pricing model fits into our abstract setting. An interpretation is as follows. A monopolist sells a service (*e.g.,* a car or vacation house rental) that can be utilized to different extents. The monopolist chooses both a menu of utilization levels and prices, as in the standard nonlinear pricing problem. The monopolist faces a cost of higher utilization, akin to the production cost in Mussa and Rosen (1978). Moreover, they must write a contract that describes what levels of utilization by the buyers are acceptable. Contractibility is costly because the monopolist has to describe the acceptable levels of utilization of the good—for example, what constitutes a unit in "good" versus "bad" condition. Our results imply that utilization is contracted upon in tiers: for instance, discrete grades of condition for a car or vacation rental. This lines up with common practice. For example, the Europcar terms of service for the United Kingdom specify discrete condition levels for car returns and corresponding fees.[22]

5 Sketch of the Proof of Theorem 1

We sketch the proof of Theorem 1 in four parts. First, we establish the existence of an optimal contractibility correspondence. Second, we bound the loss in value from set-valued perturbations of contractibility. Third, we bound the cost savings from removing contractibility. Finally, we combine these bounds to rule out infinite sets and construct an explicit bound for the optimal extent of contractibility.

5.1 Existence of Optimal Contractibility

We begin by establishing the existence of an optimal contractibility correspondence. This follows from showing that the value and cost of contractibility can be seen as continuous and lower semi-continuous functions, respectively, of the sets of self-enforcing recommendations in the Hausdorff topology. Given compactness of the set of sets of self-enforcing recommendations in the Hausdorff topology, Weierstrass' Theorem then implies existence:

[21] This force may operate in addition to the rationalization proposed by Malcomson (2013) related to repeat transactions and relational contracting.

[22] As mentioned in the terms and conditions of the rental contract (Europcar, 2024), if the front bumper of a Mini/Economy rental has a dent of less than 2cm, between 2cm and 5cm, between 5cm and 15cm, or a dent larger than 15cm, then the corresponding fees are £0, £542, £694, £738.

Lemma 1 (Existence)**.** *An optimal contractibility correspondence C^* exists.*

Proof. See Appendix A.4. □

A simpler route to this result would be to directly assume Hausdorff (lower semi-)continuity of the cost of contractibility when written as a function of the sets of self-enforcing recommendations. However, the class of examples we have developed based on costs of distinguishing are L_1 continuous by inspection and, while we prove they are Hausdorff continuous in the appropriate sense, we regard this consequence as non-obvious (see Proposition 8).

5.2 The Opportunity Cost of Coarsening Contractibility

We first bound the loss to the principal from perturbing any set of self-enforcing recommendations $\overline{D} \in \overline{\mathcal{D}}$ to remove the contractibility of some actions. The collection of feasible sets of self-enforcing recommendations $\overline{\mathcal{D}}$ and the value $\mathcal{J} : \overline{\mathcal{D}} \to \mathbb{R}$ are defined at the end of Section 3.1. We remind that $\bar{J}_{xx} = \max_{x,\theta} |J_{xx}(x,\theta)|$, $\underline{J}_{x\theta} = \min_{x,\theta} J_{x\theta}(x,\theta)$, and $\bar{f} = \max_\theta f(\theta)$.

Lemma 2. *Consider any $\overline{D} \in \overline{\mathcal{D}}$ and any $a, b \in \overline{D}$ such that $a < b$. We have:*

$$\mathcal{J}(\overline{D}) - \mathcal{J}(\overline{D} \setminus (a,b)) \leq \frac{3}{2} \frac{\bar{J}_{xx}^2 \bar{f}}{\underline{J}_{x\theta}} (b-a)^3 \tag{36}$$

Moreover, if $(a,b) \cap \overline{D} \neq \emptyset$, then there exists $c \in (a,b) \cap \overline{D}$ such that:

$$\mathcal{J}(\overline{D}) - \mathcal{J}(\overline{D} \setminus (a,b)) \leq \frac{3}{2} \frac{\bar{J}_{xx}^2 \bar{f}}{\underline{J}_{x\theta}} (b-a) \left[(c-a)^2 + (b-c)^2\right] \tag{37}$$

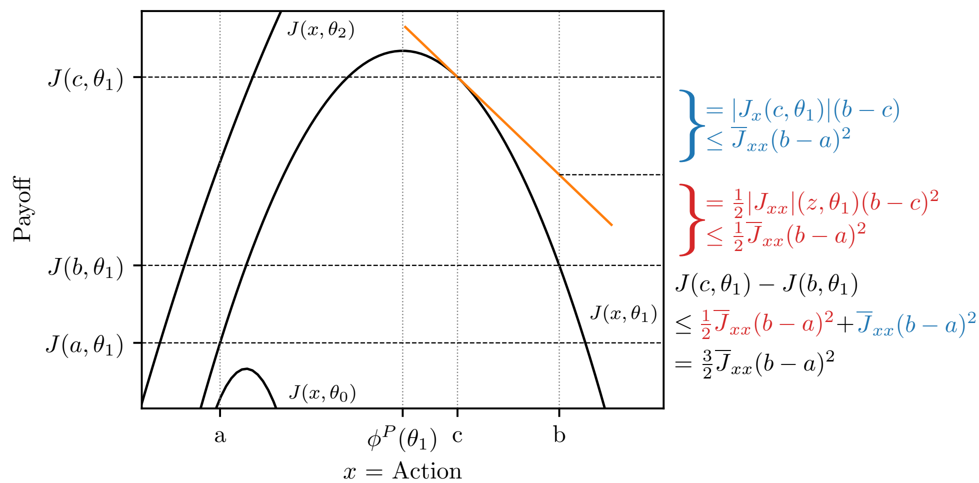
Furthermore, if $\{a, c, b\}$ are sequential, or $\overline{D} \cap (a,c) = \emptyset$ and $\overline{D} \cap (c,b) = \emptyset$, then

$$\mathcal{J}(\overline{D}) - \mathcal{J}(\overline{D} \setminus (a,b)) \leq 3 \frac{\bar{J}_{xx}^2 \bar{f}}{\underline{J}_{x\theta}} (b-a)(c-a)(b-c) \tag{38}$$

Proof. See Appendix A.5. □

The first statement says that the opportunity cost of removing all points of contractibility within an interval (a,b) is *third-order* in the length of that interval. The next two statements refine this bound when there is a third point of contractibility $c \in (a,b)$ and furthermore when the three points of interest are isolated. All three bounds share the following comparative statics: they loosen when J has higher concavity, when J has lower supermodularity, and when the type density is more concentrated.

Figure 6: Illustrating the Payoff Loss from Changing an Agent's Allocation (Lemma 2)



Note: This figure illustrates where the bound on payoffs used in the proof of Lemma 2 comes from. The black curves denote the virtual surplus function for three types $\theta_0 < \theta_1 < \theta_2$. Actions (a, b, c) are contractible (*i.e.*, among the self-enforcing recommendations) and z is some point in $[c, b]$, from Taylor's remainder theorem. The orange line is tangent to $J(\cdot, \theta_1)$ at $x = c$.

To provide intuition, we sketch the derivation of the first claim (Equation 36). Exploiting the fact that allocations conditional on any level of contractibility solve a pointwise program (see Equation 13), we write

$$\mathcal{J}(\overline{D}) - \mathcal{J}(\overline{D} \setminus (a,b)) = \int_{\Theta} (J(\phi^*(\theta), \theta) - J(\phi^{*\prime}(\theta), \theta))\, \mathrm{d}F(\theta) \tag{39}$$

where ϕ^* and $\phi^{*\prime}$ respectively denote the optimal final action functions under each level of contractibility. We next observe, using our characterization of the optimal contract (Proposition 2), that $\phi^* \neq \phi^{*\prime}$ only for types such that the actions $\overline{\phi}(\theta)$ or $\underline{\phi}(\theta)$ (from Equation 10, defined relative to $\overline{D}$), were within (a, b). The third-order bound follows from two steps: (i) showing that this set of affected types has measure proportional to $b - a$ and (ii) showing that the payoff losses for each such type are bounded by something proportional to $(b - a)^2$.

For the first step, we observe that a necessary condition for a type θ to be affected by the removal of the interval (a, b) is that $\phi^P(\theta) \in (a, b)$: that is, the principal would like (absent imperfect contractibility) to allocate these agents an action between a and b. This set of types is $\{\theta : \phi^P(\theta) \in (a, b)\}$ and has a large measure if ϕ^P is very flat (*i.e.*, nearby types map to similar actions) or if the type density is very large. We bound the inverse slope of ϕ^P by $\frac{\bar{J}_{xx}}{\underline{J}_{x\theta}}$ (by the implicit function theorem and the inverse function theorem) and the maximum type density by $\bar{f}$. Together, this contributes a term $(b - a)\frac{\bar{J}_{xx}}{\underline{J}_{x\theta}}\bar{f}$ to the bound.

For the second step, we bound the payoff loss for the principal from perturbing the allocation of any affected type θ. The calculation is visualized in Figure 6 and explained in detail below. We start by expressing $J(\cdot, \theta)$ as second-order around $\phi^*(\theta)$ using Taylor's remainder theorem. In the figure, we illustrate these calculations for a given type θ_1 such that $\phi^*(\theta_1) = c$. In this case, it is optimal under the variation to allocate this type to the action $x = b$. The second-order term in Taylor's remainder theorem is bounded above by the red term $\frac{1}{2}\overline{J}_{xx}(b-a)^2$, using the global bound for the second derivative and the fact that $b - c < b - a$. We next consider the first-order term of the quadratic representation. This has coefficient $J_x(\phi^*(\theta), \theta)$. If it were the case that $\phi^*(\theta) = \phi^P(\theta)$, this term would be zero by the envelope theorem. But, more generally, we know only that $\phi^*(\theta) \in (a, b)$ and $\phi^P(\theta) \in (a, b)$. To proceed, we apply Taylor's remainder theorem *once more* to $J_x(\cdot, \theta)$ around $\phi^P(\theta)$. Exploiting the fact that $J_x(\phi^P(\theta), \theta) = 0$, we obtain that $J_x(\phi^*(\theta), \theta)$ is linear in $\phi^*(\theta) - \phi^P(\theta)$, with a slope that is bounded above by $\overline{J}_{xx}$. This contributes the blue term in Figure 6, which is bounded above by $\overline{J}_{xx}(b-a)^2$. Putting these two bounds together, we obtain the total bound of $\frac{3}{2}\bar{J}_{xx}(b-a)^2$.

5.3 The Cost Savings of Coarsening Contractibility

We now leverage strong monotonicity of costs to understand the cost savings from removing contractibility. First, we observe that strong monotonicity implies that, for any optimal contractibility correspondence, $\underline{\delta}$ equals zero (except perhaps at $\overline{x}$):

Lemma 3. *Suppose that Γ is strongly monotone. If C^* is an optimal contractibility correspondence, then $\underline{\delta}^*(X) \subseteq \{0, \overline{x}\}$.*

Proof. See Appendix A.6. □

Strong monotonicity implies that any $\underline{\delta}$ that does not satisfy the property in Lemma 3 must yield a strictly positive cost and therefore, as the value $\mathcal{J}$ depends only on $\overline{D}$, must be suboptimal. There are two $\underline{\delta}$ functions compatible with the property in Lemma 3: $\underline{\delta} = 0$ and $\underline{\delta}(x) = \overline{x}\,\mathbb{I}[x = \overline{x}]$. As the L_1−distance between these two functions is zero, L_1−continuity of Γ implies that $\Gamma(\underline{\delta}, \overline{\delta}) = \Gamma(0, \overline{\delta})$ for any optimal contractibility correspondence. This shows that the cost component of the principal's payoff depends only on the set of self-enforcing recommendations $\overline{D}$. Thus, with some abuse of notation, we henceforth write $\Gamma(\overline{D}) = \Gamma(0, \overline{\delta})$.

We next show that strong monotonicity of Γ places lower bounds on the asymptotic marginal costs of removing contractibility.

Lemma 4 (Asymptotic Cost Savings)**.** *Suppose that* Γ *is strongly monotone with constant* $\varepsilon > 0$*. We have that:*

$$\liminf_m \frac{\Gamma(\overline{D}) - \Gamma(\overline{D} \setminus (a_m, b_m))}{(x_m - a_m)(b_m - x_m)} \geq \varepsilon \tag{40}$$

for all $\overline{D} \in \overline{\mathcal{D}}$*, accumulation points* $x \in \overline{D}$*, and sequences* $\{a_m, x_m, b_m\}_{m=1}^{\infty} \subseteq \overline{D}$ *such that* $x_m \in (a_m, b_m)$ *and* $\overline{D} \cap (a_m, b_m) \to \{x\}$*, where the limit is in the topological sense.*[23]

Proof. See Appendix A.7. □

This result formalizes the sense in which there are *second-order costs of perfect contractibility*. To illustrate this most clearly, consider an x and $\overline{D}$ such that there is perfect contractibility in a neighborhood around x, *i.e.*, $B_t(x) \subset \overline{D}$ for all sufficiently small $t > 0$, where $B_t(x)$ denotes the open ball centered at x and with radius t. In this construction, x is an (interior) accumulation point that the principal can precisely differentiate from all of its neighbors. We can take a sequence $\{t_m\}_{m=0}^{\infty}$ such that $t_m \to 0$ and construct sequences $a_m = x - t_m$ and $b_m = x + t_m$. In this case, the operation described in Lemma 4 is to remove a sequence of shrinking balls centered around x. A cost Γ satisfies Equation 40 if, in such a scenario, the cost of removing these balls is asymptotically bounded by εt_m^2.

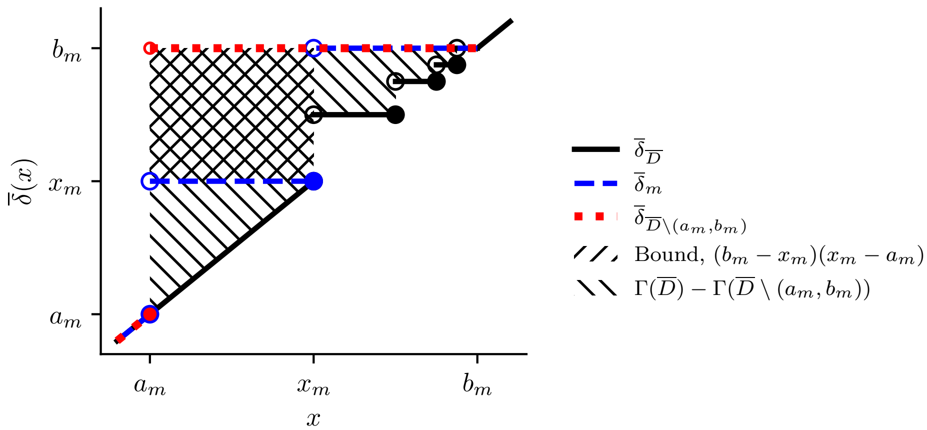
Lemma 4 generalizes this idea to also discipline the cost of precise contracting around non-interior accumulation points. For example, the set $\overline{D} = \{1 - 2^{-k}\}_{k=0}^{\infty} \cup \{1\}$ has an empty interior, but 1 is an accumulation point which the principal can distinguish from any close action $1 - 2^{-k}$, for arbitrarily large k. Similarly, if $\overline{D}$ were the Cantor set, then *all* of its elements are non-interior accumulation points.

We can give a simple intuition for why strongly monotone costs imply this asymptotic property. To illustrate this most concretely, suppose that Γ is the linear cost of distinguishing of Example 3, in which case $\varepsilon = \kappa$ and the cost coincides with the area above $\overline{\delta}$. Intuitively, removing contractibility in a ball of radius t removes the need to distinguish $2t$ actions from $2t$ other actions. This yields a cost saving of $\kappa 2t \times 2t = \kappa 4t^2$.

Figure 7 geometrically illustrates our more general bounds on marginal cost under the linear cost with $\kappa = 1$. Beginning with any $\overline{D}$ and corresponding $\overline{\delta}_{\overline{D}}$ (black solid line), we remove all self-enforcing recommendations in (a_m, b_m) to construct $\overline{\delta}_{\overline{D} \setminus (a_m, b_m)}$ (red dotted line). To derive our bound, we also construct $\overline{\delta}_m$ (blue dashed line) by deleting (a_m, b_m) but retaining x_m for some $x_m \in \overline{D}$. Our goal is to bound the cost savings from removing (a_m, b_m), equal to the area shaded in the figure with left hatches. We calculate the difference

[23] The upper topological limit of a sequence of sets $\{A_m\}_{m=1}^{\infty} \subseteq X$ is the set of points $x \in X$ such that every neighborhood intersects infinitely many sets A_m. The lower topological limit is the set of points such that every neighborhood intersects all but finitely many sets A_m. The topological limit exists if the upper and lower topological limits are equal (Definition 3.80 in Aliprantis and Border, 2006).

Figure 7: Illustrating the Cost Savings from Coarsening a Contract (Lemma 4)



Note: This figure graphically illustrates the implications of strong monotonicity (Lemma 4) for linear costs of distinguishing with $\kappa = 1$ (Example 3).

in costs between $\overline{\delta}_m$ and $\overline{\delta}_{\overline{D}\setminus(a_m,b_m)}$, equal to the rectangle which is shaded in the figure with right hatches. The rectangle has side lengths $b_m - x_m$ and $x_m - a_m$, and therefore area $(b_m - x_m)(x_m - a_m)$. Thus, $\Gamma(\overline{D}) - \Gamma(\overline{D} \setminus (a_m, b_m)) \geq (x_m - a_m)(b_m - x_m)$ and the implication of Lemma 4 holds with $\varepsilon = 1$.

5.4 Establishing Coarseness

We now combine the arguments above and establish that there exists some $K^* \in \mathbb{N}$ such that any optimal set of self-enforcing recommendations, which must exist by Lemma 1, is finite with $|\overline{D}^*| \leq K^*$. We first show that if a set of self-enforcing recommendations in infinite, then it cannot be optimal.

Lemma 5 (Suboptimality of Infinite Contracts)**.** *Suppose that* Γ *is strongly monotone. If* $\overline{D} \in \overline{\mathcal{D}}$ *is an infinite set, then it is not an optimal set of self-enforcing recommendations.*

Proof. See Appendix A.8. □

To prove this result, we leverage three core arguments that rule out the possibility that an optimal set of self-enforcing recommendations is—within some neighborhood of an accumulation point—an interval, an uncountably infinite but nowhere dense set, or a countably infinite set. The intuition for this result is most easily seen in the case of intervals. Lemma 2 established that eliminating an interval of length t of contractible outcomes has a cost that is proportional to t^3. Lemma 4 established that, under strong monotonicity, eliminating an interval of radius t has (in the limit of small t) a cost that is proportional to t^2. Thus, for small enough t, the benefit of eliminating contractability exceeds the cost of keeping it. Therefore, an optimal $\overline{D}$ cannot contain an interval. Similar variational arguments that leverage different implications of Lemmas 2 and 4 rule out uncountably infinite but nowhere dense sets and countably infinite sets.

To extend this argument to rule out any infinite set, we can place any such set into one of four mutually exclusive categories: a perfect set that is somewhere dense, a perfect set that is nowhere dense, a non-perfect set that is uncountably infinite, and a non-perfect set that is countably infinite. By applying the Cantor-Bendixson theorem, we can show that each of these infinite sets must contain a neighborhood within which the set of self-enforcing recommendations reduces to one of the three cases from the previous paragraph. This yields the conclusion that an optimal set of self-enforcing recommendations cannot be infinite.

We now leverage the finiteness of an optimal set of self-enforcing recommendations to derive an explicit upper bound on the number of such recommendations.

Lemma 6 (The Fineness Bound)**.** *If* Γ *is strongly monotone with constant* $\varepsilon > 0$*, then any optimal set of self-enforcing recommendations satisfies* $|\overline{D}^*| \leq 2 + \left\lfloor 6\frac{\overline{x}\bar{J}_{xx}^2\bar{f}}{\varepsilon \underline{J}_{x\theta}} \right\rfloor$.

Proof. See Appendix A.9. □

We prove this by using our explicit bound on the payoff gains from more complete contracts from Lemma 2. If more than this many actions were contractible, we show that eliminating at least one self-enforcing recommendation would be payoff improving.

Finally, we translate this result into the form stated in Theorem 1. Observe that if $\overline{D}^*$ is finite, then so too is $\overline{\delta}^*(X)$. We have also already shown that $\underline{\delta}^*(X) \subseteq \{0, \overline{x}\} \subseteq \overline{\delta}^*(X)$. Thus, since $|C(X)| = |\overline{\delta}^*(X)|$ in this case, Lemma 6 implies the bound of Theorem 1.

6 Conclusion

In this paper, we introduce a model of contractibility design. Our analysis has two premises. The first is that contracts are only enforceable to the extent that the principal can prove that the agent deviated from the terms of the contract. The second is that the codification and generation of evidence that can be used to provide such proof entails front-end costs. We show that a large class of front-end costs satisfy a monotonicity property that we call strong monotonicity. Under this property, optimal contracts are coarse. We show how the agents' preferences, the principal's preferences, and the incompleteness of information affect the structure and potential coarseness of contracts. We argue that our model generates insights into the nature of employment contracts, procurement contracts, and monopoly pricing.

We conclude this section by mentioning other settings in which our general framework, methods, and results can be applied. A first class of applications includes many settings in which (monotone) partitions of information are the economic object of interest, like certain models of costly information acquisition and costly certification (*e.g.,* Gul, Pesendorfer, and Strzalecki, 2017; Ellis, 2018; Zapechelnyuk, 2020). Monotone partitions are nested as the special case of regular contractibility correspondences when Transitivity is strengthened to the following Symmetry property: for all $x, y \in X$, if $x \in C(y)$, then $C(x) = C(y)$. When the principal is restricted to choose a monotone partition in this new feasible set, our main coarseness result (Theorem 1) still holds, with a possibly different finite bound. Mapped to the problems described above, our analysis gives conditions under which optimal information structures take the striking form of a finite partition.

An additional direction for further research is applying our analysis to models of costly persuasion and delegation. These settings, when restricted to deterministic and monotone mechanisms, also feature monotone partitions (equivalently, closed sets) as the choice variable (Kolotilin and Zapechelnyuk, 2025). However, unlike those described above, these settings feature misalignment in incentives between the two parties that would complicate the mapping from partitions to the designer's value (*i.e.,* the analog of our Equation 13). Analyzing such a model would require new bounds on the loss in value from perturbations of contractibility (our Lemma 2), but the general approach of our analysis would apply.

A second class of related problems draws from the literature on mechanism design with evidence about agents' types (see *e.g.,* Green and Laffont, 1986; Hart, Kremer, and Perry, 2017), rather than their actions. Part of this literature studies the problem of designing optimal mechanisms given a fixed correspondence describing the set of types $M(\theta)$ that each type $\theta \in \Theta$ can mimic. For example, Krähmer and Strausz (2024) study a model in which agents can only mimic agents of a higher type, *i.e.,* $M(\theta) = [\theta, 1]$, and Sher and Vohra (2015) study a discrete-type model with a more general correspondence. It would be interesting to study the *costly design* of such reporting constraints and the resulting implications for optimal contracts in future research.

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Appendix to Contractibility Design by Corrao, Flynn, and Sastry

A Proofs

A.1 Proof of Proposition 1

(If). Let $C : X \rightrightarrows X$ be a regular contractibility correspondence and define $\underline{\delta}(y) = \min C(y)$ and $\overline{\delta}(y) = \max C(y)$ for all $y \in X$. By the fact that C is closed-valued, $\overline{\delta}$ and $\underline{\delta}$ exist. By monotonicity, we have that $\underline{\delta}$ and $\overline{\delta}$ are increasing functions. By reflexivity, we know that $y \geq \underline{\delta}(y)$ and $y \leq \overline{\delta}(y)$ for all y. Moreover, by Lemma 17.29 in Aliprantis and Border (2006), $\overline{\delta}$ is lower semi-continuous and $\underline{\delta}$ is upper semi-continuous.

We now show that $C(y) = [\underline{\delta}(y), \overline{\delta}(y)]$. Assume by contradiction there exists some $y \in X$ and $x \in [\underline{\delta}(y), \overline{\delta}(y)]$ such that $x \notin C(y)$. Consider first the case where $x < y$. By the definition of $\underline{\delta}$, $\underline{\delta}(y) \in C(y)$ and $\underline{\delta}(y) < x$. As $x < y$, by monotonicity, we have that $C(x) \leq_{SSO} C(y)$. Thus, as $x \in C(x)$ and $\underline{\delta}(y) \in C(y)$, we know that $\max\{x, \underline{\delta}(y)\} = x \in C(y)$. This is a contradiction. Consider now the case where $y < x$. Again, $\overline{\delta}(y) \in C(y)$ and $x < \overline{\delta}(y)$. By monotonicity, we have that $\min\{x, \overline{\delta}(y)\} = x \in C(y)$. This is a contradiction.

We next show parts (ii), (iii), and (iv). Fix $x, y \in X$ and assume that $x \in [\underline{\delta}(y), \overline{\delta}(y))$, which implies $x \in C(y)$. We start with part (ii), and mirror the argument for part (iii). Suppose $x < y$. As C is monotone, we know that $\underline{\delta}(x) \leq \underline{\delta}(y)$. Suppose by contradiction that $\underline{\delta}(x) < \underline{\delta}(y)$. But then, given the other properties of δ, for all $z \in (\underline{\delta}(x), \underline{\delta}(y))$ we would have that $z \in C(x)$ but $z \notin C(y)$, which contradicts transitivity. For part (iii), consider the same scenario but reversed. Suppose $x > y$. As C is monotone, we know that $\overline{\delta}(x) \geq \overline{\delta}(y)$. Imagine this held at strict inequality. Then there would exist $z \in (\overline{\delta}(y), \overline{\delta}(x))$ such that $z \in C(y)$ and $z \notin C(x)$, while $y \in C(x)$. This violates transitivity. It is immediate that $\overline{\delta}(0) = 0$ by excludability as $C(0) = \{0\}$.

(Only If). Fix $\underline{\delta}$ and $\overline{\delta}$ with the properties in Proposition 1. We show that $C(y) = [\underline{\delta}(y), \overline{\delta}(y)]$ is regular. C is reflexive because of (i), closed by construction, and monotone because $\underline{\delta}, \overline{\delta}$ are monotone. To show transitivity, consider $x \in C(y)$ and, first, the case $x < y$. From (ii), we have $\underline{\delta}(x) = \underline{\delta}(y)$. Moreover, from monotonicity, $\overline{\delta}(x) \leq \overline{\delta}(y)$. Therefore, $C(x) \subseteq C(y)$. Next, consider the case where $x > y$. From (iii), we have $\overline{\delta}(x) = \overline{\delta}(y)$. Moreover, from monotonicity, $\underline{\delta}(x) \geq \underline{\delta}(y)$. Therefore, $C(x) \subseteq C(y)$. Moreover, if $x = y$, clearly $C(x) \subseteq C(y)$. Given that these arguments hold for any x, this shows transitivity. Finally, as $\overline{\delta}(0) = \underline{\delta}(0)$, we have that $C(0) = \{0\}$, establishing excludability.

A.2 Proof of Proposition 2

We prove the result in three parts. First, we present a characterization of implementable allocations. Second, we use this characterization to derive the principal's control problem. Third, we solve this control problem for the optimal contract.

Part 1: Implementation

We begin by establishing a general taxation principle with partial contractibility. Given a regular contractibility correspondence C, we say that $T : X \to \bar{\mathbb{R}}$ is monotone with respect to C if $T(y) \geq T(x)$ for all $x, y \in X$ such that $x \in C(y)$. We now show monotonicity of the tariff with respect to C is necessary and sufficient for implementability (Definition 4).

Lemma 7 (C-Monotone Taxation Principle)**.** *Fix a regular contractibility correspondence C. A final action function ϕ is implementable given C if and only if there exists a tariff $T : X \to \bar{\mathbb{R}}$ that is monotone with respect to C and such that:*

$$\phi(\theta) \in \arg\max_{x \in X} \{u(x, \theta) - T(x)\} \tag{41}$$

and $u(\phi(\theta), \theta) - T(\phi(\theta)) \geq 0$ for all $\theta \in \Theta$. In this case, ϕ is supported by $\xi = \phi$ and T.

Proof. **(Only if)** We begin by proving the necessity of the existence of a monotone tariff with respect to C. Suppose that ϕ is implementable. It follows that there exists (ξ, T) that support ϕ. In particular, observe that (O) implies that $\phi(\theta) \in C(\xi(\theta))$ for all $\theta \in \Theta$. Next define $\hat{T} : X \to \bar{\mathbb{R}}$ as:

$$\hat{T}(x) = \inf_{y \in X} \{T(y) : x \in C(y)\} \tag{42}$$

We next show that ϕ is also supported by $(\phi, \hat{T})$. By (O) of (ϕ, ξ, T), we have

$$u(\phi(\theta), \theta) \geq u(x, \theta) \tag{43}$$

for all $x \in C(\phi(\theta)) \subseteq C(\xi(\theta))$ (by transitivity) and for all $\theta \in \Theta$, yielding (O) of $(\phi, \phi, \hat{T})$. By (IR) of (ϕ, ξ, T) and the definition of $\hat{T}$, we have

$$u(\phi(\theta), \theta) - \hat{T}(\phi(\theta)) \geq u(\phi(\theta), \theta) - T(\xi(\theta)) \geq 0 \tag{44}$$

for all $\theta \in \Theta$, yielding (IR) of $(\phi, \phi, \hat{T})$. Next, assume toward a contradiction that $(\phi, \phi, \hat{T})$ does not satisfy (IC), that is, there exist $\theta \in \Theta$ and $y \in X$ such that

$$\max_{x \in C(y)} u(x, \theta) - \hat{T}(y) > u(\phi(\theta), \theta) - \hat{T}(\phi(\theta)) \tag{45}$$

By the definition of $\hat{T}$, there exists a sequence $z_n \in X$ such that $y \in C(z_n)$ for all n and $T(z_n) \downarrow \hat{T}(y)$. Thus, there exists n large enough such that

$$\begin{aligned} &\max_{x \in C(z_n)} u(x,\theta) - T(z_n) \geq \max_{x \in C(y)} u(x,\theta) - T(z_n) \\ &> u(\phi(\theta),\theta) - \hat{T}(\phi(\theta)) \geq u(\phi(\theta),\theta) - T(\xi(\theta)) \\ &= \max_{x \in C(\xi(\theta))} u(x,\theta) - T(\xi(\theta)) \end{aligned} \tag{46}$$

The first inequality follows from $C(y) \subseteq C(z_n)$ since $y \in C(z_n)$. The second strict inequality follows from Equation 45 and the fact that $T(z_n) \downarrow \hat{T}(y)$. The third inequality follows from the construction of $\hat{T}$. The final equality follows as (ϕ, ξ, T) satisfies (O). However, the previous inequality yields a contradiction of (IC) of (ϕ, ξ, T), proving that $(\phi, \phi, \hat{T})$ satisfies (IC). This shows that $(\phi, \phi, \hat{T})$ is implementable, hence that Equation 41 holds and that $u(\phi(\theta), \theta) - T(\phi(\theta)) \geq 0$ for all $\theta \in \Theta$.

Finally, we argue that $\hat{T}$ is monotone with respect to C. Fix $x, y \in X$ such that $y \in C(x)$. By Transitivity of C we have

$$\{\hat{x} \in X : x \in C(\hat{x})\} \subseteq \{\hat{x} \in X : y \in C(\hat{x})\} \tag{47}$$

yielding that $\hat{T}(y) \leq \hat{T}(x)$, as desired.

(If) We now establish sufficiency. Suppose that there exists a tarrif $T : X \to \bar{\mathbb{R}}$ that is monotone with respect to C and such that Equation 41 holds and $u(\phi(\theta), \theta) - T(\phi(\theta)) \geq 0$ for all $\theta \in \Theta$. We will show that (ϕ, ϕ, T) is implementable. (IR) is immediately satisfied. Next, we show that (IC) is satisfied. Suppose, toward a contradiction, that it were not. That is, there exist $\theta \in \Theta$, $y \in X$, and $x \in C(y)$ such that

$$u(x,\theta) - T(y) > \max_{\hat{x} \in C(\phi(\theta))} u(\hat{x},\theta) - T(\phi(\theta)) \geq u(\phi(\theta),\theta) - T(\phi(\theta)) \tag{48}$$

But then, we have the following contradiction of monotonicity of T in C:

$$u(x,\theta) - T(y) > u(\phi(\theta),\theta) - T(\phi(\theta)) \geq u(x,\theta) - T(x) \implies T(x) > T(y) \tag{49}$$

where the second inequality uses the fact that $\phi(\theta)$ solves the program in Equation 41. Finally, we show that (O) is satisfied. Toward a contradiction, assume that there exists $\theta \in \Theta$ and $x \in C(\phi(\theta))$ such that:

$$u(x,\theta) > u(\phi(\theta),\theta) \tag{50}$$

However, by monotonicity of T in C, we know that $T(\phi(\theta)) \geq T(x)$. Thus,

$$u(x,\theta) - T(x) > u(\phi(\theta),\theta) - T(\phi(\theta)) \tag{51}$$

yielding a contradiction to IC, which we just showed. This proves sufficiency.

Finally, the fact that any implementable final action function can be implemented as part of an allocation (ϕ, ϕ, T) follows by the construction in the necessity part of our proof. □

With this taxation principle in hand, we now characterize implementation:

Lemma 8 (Implementation). *A final action function ϕ is implementable under $C = [\underline{\delta}, \overline{\delta}]$, with self-enforcing recommendation sets $\underline{D} = \underline{\delta}(X)$ and $\overline{D} = \overline{\delta}(X)$, if and only if it is monotone increasing and such that: (i) if agent preferences are monotone increasing, then $\phi(\Theta) \subseteq \overline{D}$, (ii) if preferences are monotone decreasing, then $\phi(\Theta) \subseteq \underline{D}$. Moreover, ϕ is supported by $\xi = \phi$ and tariff:*

$$T(x) = T(0) + u(x, \phi^{-1}(x)) - \int_0^{\phi^{-1}(x)} u_\theta(\phi(s), s)\, \mathrm{d}s \tag{52}$$

where $\phi^{-1}(s) = \inf\{\theta \in \Theta : \phi(\theta) \geq s\}$.

Proof. **(Only If for First Part)** If ϕ is implementable, then there exists (ξ, T) that support ϕ. By Lemma 7, we may take that $\xi = \phi$. By (IC) and Lemma 7, there exists a transfer function $t : \Theta \to \mathbb{R}$ given by $t(\theta) = T(\phi(\theta))$ such that $u(\phi(\theta), \theta) - t(\theta) \geq u(\phi(\theta'), \theta) - t(\theta')$ for all $\theta, \theta' \in \Theta$. As u is strictly single-crossing, Proposition 1 in Rochet (1987) then implies that ϕ is monotone. Without loss of generality, consider the case with monotone increasing preferences and toward a contradiction suppose that $\phi(\theta) \notin \overline{D}$. Deviating to $\overline{\delta}(\phi(\theta)) > \phi(\theta)$ is a strict improvement for the agent. Thus, if ϕ is implementable, then it is monotone, and $\phi(\Theta) \subseteq \overline{D}$ (or $\phi(\Theta) \subseteq \underline{D}$ with montone decreasing preferences) holds.

(If For First Part) Without loss of generality, we again prove this part for the case with monotone increasing preferences. Now suppose that $\phi(\theta) \in \overline{D}$ holds for all $\theta \in \Theta$ and ϕ is monotone increasing. Define the function $t : \Theta \to \mathbb{R}$ as

$$t(\theta) = Z + u(\phi(\theta), \theta) - \int_0^{\theta} u_\theta(\phi(s), s)\, \mathrm{d}s \tag{53}$$

for some $Z \leq 0$, and the tariff $T : X \to \overline{\mathbb{R}}$ as

$$T(x) = \inf_{\theta' \in \Theta} \{t(\theta') : x \in C(\phi(\theta'))\} \tag{54}$$

Fix $x, y \in X$ such that $x \in C(y)$. By Transitivity, for all $\theta \in \Theta$, if $y \in C(\phi(\theta))$, then $x \in C(\phi(\theta))$. This shows that

$$\{\theta \in \Theta : y \in C(\phi(\theta))\} \subseteq \{\theta \in \Theta : x \in C(\phi(\theta))\} \tag{55}$$

Therefore, applying the construction of T, $T(y) \geq T(x)$. Thus, T is monotone with respect to C.

As T is monotone with respect to C, if we can show that $\phi(\theta) \in \arg\max_{x \in X}\{u(x, \theta) - T(x)\}$ and $u(\phi(\theta), \theta) - T(\phi(\theta)) \geq 0$, then we have shown by Lemma 7 that ϕ is implementable.

We start with the second condition. For every $\theta \in \Theta$, we have

$$u(\phi(\theta), \theta) - T(\phi(\theta)) \geq u(\phi(\theta), \theta) - t(\theta) = \int_0^\theta u_\theta(\phi(s), s)\, \mathrm{d}s - Z \tag{56}$$

Note that the right-hand side of this last equation is monotone increasing in θ since it is continuously differentiable with derivative $u_\theta(\phi(\theta), \theta) = \int_0^{\phi(\theta)} u_{x\theta}(z, \theta)\, \mathrm{d}z \geq 0$ for all $\theta \in \Theta$, owing to the fact that u is supermodular. Given that $Z \leq 0$, we have that $u(\phi(\theta), \theta) - T(\phi(\theta)) \geq 0$ for all $\theta \in \Theta$.

We are left to prove that (ϕ, T) satisfy Equation 41. We first prove that, for all $\theta, \theta' \in \Theta$:

$$u(\phi(\theta), \theta) - t(\theta) \geq \max_{x \in C(\phi(\theta'))} u(x, \theta) - t(\theta') \tag{57}$$

This is a variation of the standard reporting problem under consumption function ϕ and transfers t, where each agent, on top of misreporting their type, can also consume everything allowed by C. Violations of this condition can take two forms. First, an agent of type θ could report type θ' and consume $x = \phi(\theta')$. We call this a single deviation. Second, an agent of type θ could report type θ' and consume $x \in C(\phi(\theta')) \setminus \{\phi(\theta')\}$. We call this a double deviation. Under our construction of transfers t and monotonicity of ϕ, by a standard mechanism-design argument, there is no strict gain to any agent of reporting θ' and consuming $x = \phi(\theta')$. Thus, there are no profitable single deviations under (ϕ, t).

We now must rule out double deviations. Suppose that θ imitates θ' and plans to take final action $x \neq \phi(\theta')$. As $\phi(\theta') \in \overline{D}$ (in the monotone increasing case), $x < \phi(\theta')$. But in that case, simply taking action $\phi(\theta')$ is better. But then this is a single deviation, which we have ruled out. The same logic applies in the monotone decreasing case.

To derive the tariff, we can simply set $T(x) = t(\phi^{-1}(x))$. This yields the claimed formula. □

Part 2: Control Problem

We now use this characterization of implementation to turn the principal's problem into an optimal control problem:

Lemma 9. *When agents have monotone increasing preferences, any optimal final action function solves:*

$$\begin{aligned} \mathcal{J}(\overline{D}) = \max_{\phi} \quad & \int_{\Theta} J(\phi(\theta), \theta) \, \mathrm{d}F(\theta) \\ s.t. \quad & \phi(\theta') \geq \phi(\theta), \phi(\theta) \in \overline{D}, \quad \theta, \theta' \in \Theta : \theta' \geq \theta \end{aligned} \tag{58}$$

When agents have monotone decreasing preferences, replace $\overline{D}$ with $\underline{D}$.

Proof. We begin by eliminating the proposed allocation and transfers from the objective function of the principal. From the proof of Lemma 8, we have that transfers for any incentive compatible triple (ξ, ϕ, t) are given by:

$$t(\theta) = Z + u(\phi(\theta), \theta) - \int_0^{\theta} u_{\theta}(\phi(s), s) \, \mathrm{d}s \tag{59}$$

for some constant $Z \in \mathbb{R}$. Thus, any ξ that supports ϕ leads to the same principal payoff and can therefore be made equal to ϕ without loss of optimality. Moreover, we know that ϕ being incentive compatible is equivalent to ϕ being monotone increasing and $\phi(\theta) \in \overline{D}$.

Plugging in the expression (59), we can simplify the expression for the principal's total transfer revenue as the following:

$$\begin{aligned} \int_{\Theta} t(\theta) \, \mathrm{d}F(\theta) &= \int_{\Theta} \left(Z + u(\phi(\theta), \theta) - \int_0^{\theta} u_{\theta}(\phi(s), s) \, \mathrm{d}s \right) \mathrm{d}F(\theta) \\ &= \int_{\Theta} (Z + u(\phi(\theta), \theta)) \, \mathrm{d}F(\theta) - \int_0^1 \int_0^{\theta} u_{\theta}(\phi(s), s) \, \mathrm{d}s \, \mathrm{d}F(\theta) \end{aligned} \tag{60}$$

Using this expression for total transfer revenue, and the characterization of implementation from Lemma 8, we write the principal's problem as

$$\begin{aligned} \max_{\phi, Z} \quad & \int_{\Theta} \left(\pi(\phi(\theta), \theta) + Z + u(\phi(\theta), \theta) - \int_0^{\theta} u_{\theta}(\phi(s), s) \, \mathrm{d}s \right) \mathrm{d}F(\theta) \\ \text{s.t.} \quad & \phi(\theta') \geq \phi(\theta), \ \phi(\theta) \in \overline{D} \quad \forall \theta, \theta' \in \Theta : \theta' \geq \theta \\ & u(\phi(\theta), \theta) - \left(Z + u(\phi(\theta), \theta) - \int_0^{\theta} u_{\theta}(\phi(s), s) \, \mathrm{d}s \right) \geq 0 \quad \forall \theta \in \Theta \end{aligned} \tag{61}$$

We further simplify this by applying integration by parts on the double integral of $u_{\theta}(\phi(s), s)$

over θ and s:

$$\begin{aligned}\int_0^1 \int_0^\theta u_\theta(\phi(s), s)\,\mathrm{d}s\,\mathrm{d}F(\theta) &= \left[F(\theta)\int_0^\theta u_\theta(\phi(s); s)\,\mathrm{d}s\right]_0^1 - \int_0^1 F(\theta)u_\theta(\phi(\theta),\theta)\,\mathrm{d}\theta \\ &= \int_0^1 (1-F(\theta))u_\theta(\phi(\theta),\theta)\,\mathrm{d}\theta \\ &= \int_0^1 \frac{(1-F(\theta))}{f(\theta)}u_\theta(\phi(\theta),\theta)\,\mathrm{d}F(\theta)\end{aligned} \tag{62}$$

Plugging into the principal's objective, we find that the principal solves:

$$\begin{aligned} &\max_{\phi, Z} \int_\Theta (J(\phi(\theta)) + Z)\,\mathrm{d}F(\theta) \\ &\text{s.t.} \quad \phi(\theta') \geq \phi(\theta),\ \phi(\theta) \in \overline{D} \quad \forall \theta, \theta' \in \Theta : \theta' \geq \theta \\ &\quad u(\phi(\theta),\theta) - \left(Z + u(\phi(\theta),\theta) - \int_0^\theta u_\theta(\phi(s), s)\,\mathrm{d}s\right) \geq 0 \quad \forall \theta \in \Theta \end{aligned} \tag{63}$$

It follows that it is optimal to set $Z \in \mathbb{R}$ as large as possible such that:

$$V(\theta) = u(\phi(\theta),\theta) - \left(Z + u(\phi(\theta),\theta) - \int_0^\theta u_\theta(\phi(s), s)\,\mathrm{d}s\right) \geq 0 \quad \forall \theta \in \Theta \tag{64}$$

We know that $V'(\theta) = u_\theta(\phi(\theta),\theta) \geq 0$ as we have already shown that $u(x, \cdot)$ is monotone over Θ. Thus, the tightest such constraint occurs when $\theta = 0$. Hence, the maximal Z must satisfy:

$$V(0) = -Z \geq 0 \tag{65}$$

This implies that Z is optimally 0 and ensures that the (IR) constraint holds for all types. Hence, the principal's program is:

$$\begin{aligned} &\max_{\phi} \int_\Theta J(\phi(\theta),\theta)\,\mathrm{d}F(\theta) \\ &\text{s.t.} \quad \phi(\theta') \geq \phi(\theta),\ \phi(\theta) \in \overline{D} \quad \forall \theta, \theta' \in \Theta : \theta' \geq \theta \end{aligned} \tag{66}$$

This completes the proof. □

Part 3: The Optimal Contract

We first solve the pointwise problem in the control problem from Lemma 9 and then verify that this solution is monotone. The pointwise problem is $\max_{x \in \overline{D}} J(\phi(\theta),\theta)$, where the maximum exists as J is continuous and $\overline{D}$ is compact. As J is strictly quasi-concave, this maximum is either $\overline{\phi}(\theta)$ or $\underline{\phi}(\theta)$. Define $\Delta J(\theta) = J(\overline{\phi}(\theta), \theta) - J(\underline{\phi}(\theta), \theta)$ for all $\theta \in \Theta$. When $\Delta J(\theta) > 0$, it is $\overline{\phi}(\theta)$. When $\Delta J(\theta) < 0$, it is $\underline{\phi}(\theta)$. When $\Delta J(\theta) = 0$, either is optimal. Thus, if it is monotone, the claimed solution is optimal (as it is supported on $\overline{D}$). Moreover, this pointwise solution is monotone by Theorem 4′ in Milgrom and Shannon (1994). The claim that $\xi^* = \phi^*$ and the formula for the optimal tariff follow immediately from applying Lemma 8.

A.3 Proof of Proposition 3

We first prove the result for an arbitrary uncertain cost of distinguishing actions defined by any compactly supported probability measure $\mu \in \Delta(\mathcal{G})$, a CARA coefficient $\lambda \in (-\infty, \infty)$. Observe that the result for the standard cost of distinguishing for any $g \in \mathcal{G}$ immediately follows by taking μ such that $\operatorname{supp}\mu = \{g\}$.

For all $g \in \mathcal{G}$, define $\underline{g} = \min_{(x,y)\in X^2} g(x, y) > 0$ and $\overline{g} = \max_{(x,y)\in X^2} g(x, y) > 0$. Both the minimum and maximum are attained and strictly positive because each g is a strictly positive and continuous function defined over the compact set X^2. Fix $C', C \in \mathcal{C}$ such that $C' \subseteq C$. By the mean-value theorem applied to $\frac{1}{\lambda}\log(k)$, there exists

$$\hat{k}(C', C) \in \left[\int_{\mathcal{G}} \exp\left(\lambda\Gamma^g(C)\right) \mathrm{d}\mu\left(g\right), \int_{\mathcal{G}} \exp\left(\lambda\Gamma^g(C')\right) \mathrm{d}\mu\left(g\right)\right] \tag{67}$$

such that

$$\Gamma^{\mu,\lambda}(C') - \Gamma^{\mu,\lambda}(C) = \frac{1}{\lambda\hat{k}(C', C)} \left(\int_{\mathcal{G}} \left[\exp\left(\lambda\Gamma^g(C')\right) - \exp\left(\lambda\Gamma^g(C)\right)\right] \mathrm{d}\mu\left(g\right)\right) \tag{68}$$

Next, by applying the mean-value theorem to $\exp(\lambda r)$, for every $g \in \operatorname{supp}\mu$, there exists $\hat{r}(C', C, g) \in [\Gamma^g(C), \Gamma^g(C')]$ such that

$$\exp\left(\lambda\Gamma^g(C')\right) - \exp\left(\lambda\Gamma^g(C)\right) = \lambda\exp(\lambda\hat{r}(C', C, g))\left(\Gamma^g(C') - \Gamma^g(C)\right) \tag{69}$$

Another application of the mean-value theorem yields that, for every $g \in \operatorname{supp}\mu$,

$$\begin{aligned}
\Gamma^g(C') - \Gamma^g(C) &= \int_0^{\overline{x}} \left[G(\underline{\delta}'(y), y) - G(\underline{\delta}(y), y)\right] \mathrm{d}y + \int_0^{\overline{x}} \left[G(\overline{\delta}(y), y) - G(\overline{\delta}'(y), y)\right] \mathrm{d}y && (70)\\
&= \int_0^{\overline{x}} g(\hat{\omega}(y), y) \left[\underline{\delta}'(y) - \underline{\delta}(y)\right] \mathrm{d}y + \int_0^{\overline{x}} g(\tilde{\omega}(y), y) \left[\overline{\delta}(y) - \overline{\delta}'(y)\right] \mathrm{d}y && (71)\\
&\geq \underline{g}L(C \setminus C') && (72)
\end{aligned}$$

where $\hat{\omega}(y) \in [\underline{\delta}(y), \underline{\delta}'(y)]$ and $\tilde{\omega}(y) \in [\overline{\delta}'(y), \overline{\delta}(y)]$ for every $y \in X$. By combining these last three steps together we have

$$\Gamma^{\mu,\lambda}(C') - \Gamma^{\mu,\lambda}(C) \geq \int_{\mathcal{G}} \underline{g} \frac{\exp(\lambda \hat{r}(C', C, g))}{\hat{k}(C', C)} \mathrm{d}\mu(g) L(C \setminus C') \tag{73}$$

$$\geq \frac{\int_{\mathcal{G}} \underline{g} \mathrm{d}\mu(g)}{\int_{\mathcal{G}} \exp\left(\lambda \overline{g}\right) \mathrm{d}\mu\left(g\right)} L(C \setminus C') \tag{74}$$

where the second equality follows because $\exp(\lambda \hat{r}(C', C, g)) \geq 1$ and $\hat{k}(C', C) \leq \int_{\mathcal{G}} \exp\left(\lambda \overline{g}\right) \mathrm{d}\mu\left(g\right)$. We can therefore set:

$$\varepsilon = \frac{\int_{\mathcal{G}} \underline{g} \mathrm{d}\mu(g)}{\int_{\mathcal{G}} \exp\left(\lambda \overline{g}\right) \mathrm{d}\mu\left(g\right)} > 0 \tag{75}$$

where the inequality follows by the facts that $0 < \underline{g} \leq \overline{g} < \infty$ for all $g \in \mathcal{G}$ and that the support of μ is compact. With this, we have shown that $\Gamma^{\mu,\lambda}$ is strongly monotone for all compactly supported $\mu \in \Delta(\mathcal{G})$ and $\lambda \in (-\infty, \infty)$.

It remains to establish strong monotonicity for $\lambda \in \{-\infty, \infty\}$. Consider the case where $\lambda = \infty$. We have that:

$$\begin{aligned} \Gamma^{\mu,\infty}(C') - \Gamma^{\mu,\infty}(C) &= \max_{g \in \operatorname{supp} \mu} \Gamma^{g}(C') - \max_{g \in \operatorname{supp} \mu} \Gamma^{g}(C) \\ &\geq \Gamma^{g_C}(C') - \Gamma^{g_C}(C) \end{aligned} \tag{76}$$

for any $g_C \in \arg\max_{g \in \operatorname{supp} \mu} \Gamma^g(C)$. We have already shown that Γ^g is strongly monotone for all $g \in \mathcal{G}$ by Equation 70. Thus, $\Gamma^{\mu,\infty}$ is strongly monotone with $\varepsilon = \min_{g \in \operatorname{supp} \mu} \underline{g} > 0$, where the strict inequality follows by continuity of the map $g \mapsto \underline{g}$ (which follows by an application of Berge's theorem) and compactness of $\operatorname{supp} \mu$. The case with $\lambda = -\infty$ follows by a symmetric argument (i.e., take $g_{C'}$ instead).

A.4 Proof of Lemma 1

By Proposition 1 and Lemma 11 we can uniquely represent $C \in \mathcal{C}$ by $(\underline{D}, \overline{D}) \in \underline{\mathcal{D}} \times \overline{\mathcal{D}}$ (and vice-versa), where $\underline{\mathcal{D}}$ is the collection of closed subsets of X that contain 0. We write the regular contractibility correspondence generated by the sets of self-enforcing recommendations be $C_{(\underline{D},\overline{D})}$. We endow $\underline{\mathcal{D}} \times \overline{\mathcal{D}}$ with the product topology induced by the Hausdorff topology on each collection of sets. With this, using the characterization of the value of contractibility from Proposition 2, we observe that Problem 8 is equivalent to the problem:

$$\sup_{(\underline{D},\overline{D}) \in \underline{\mathcal{D}} \times \overline{\mathcal{D}}} \mathcal{J}(\overline{D}) - \Gamma\left(C_{(\underline{D},\overline{D})}\right) \tag{77}$$

To show existence, it now suffices to argue that (i) the domain $\underline{\mathcal{D}} \times \overline{\mathcal{D}}$ is compact in the Hausdorff topology, and (ii) $\Gamma(C_{(\cdot,\cdot)})$ is lower semi-continuous in the Hausdorff topology, (iii) $\mathcal{J}$ is continuous in the Hausdorff topology. The result then follows from Weierstrass' Theorem.

In Lemma 13, we show that $\underline{\mathcal{D}} \times \overline{\mathcal{D}}$ is compact in the Hausdorff topology. In Proposition 8, we leverage L_1 semi-continuity of Γ in $(\underline{\delta}, \overline{\delta})$ to show that $\Gamma(C_{(\cdot,\cdot)})$ is lower semi-continuous in $(\underline{D}, \overline{D})$ in the Hausdorff topology. We now show continuity of $\mathcal{J}$. By Lemma 9 and since $J(x, \theta)$ is strictly supermodular, we have

$$\mathcal{J}(\overline{D}) = \int_{\Theta} \tilde{\mathcal{J}}(\overline{D}, \theta) \, \mathrm{d}F(\theta) \tag{78}$$

where we define

$$\tilde{\mathcal{J}}(\overline{D}, \theta) = \max_{x \in \overline{D}} J(x, \theta) \tag{79}$$

for all $\theta \in \Theta$ and $\overline{D} \in \overline{\mathcal{D}}_B$. By Berge's maximum theorem, it follows that the map $(\overline{D}, \theta) \mapsto \tilde{\mathcal{J}}(\overline{D}, \theta)$ is continuous. This, the fact that $\tilde{\mathcal{J}}$ is bounded, and the dominated converge theorem together imply that $\overline{D} \mapsto \mathcal{J}(\overline{D})$ is continuous in the Hausdorff topology.

A.5 Proof of Lemma 2

Let ϕ^* denote the optimal allocation under $\overline{D}$ and $\phi^{*'}$ denote the optimal allocation under $\overline{D}' = \overline{D} \setminus (a, b)$, as defined in Proposition 2. By Lemma 9, the difference in values under these contractibility correspondences is

$$\mathcal{J}(\overline{D}) - \mathcal{J}(\overline{D}') = \int_0^1 (J(\phi^*(\theta), \theta) - J(\phi^{*'}(\theta), \theta)) \, \mathrm{d}F(\theta) \tag{80}$$

First, we observe that $\phi^*(\theta) \neq \phi^{*'}(\theta)$ only if $\phi^*(\theta) \in (a, b)$. We denote the set of types who receive such allocations by $\Theta(a, b) = \{\theta \in \Theta : \phi^*(\theta) \in (a, b)\}$. As ϕ^* is monotone, this is an interval. If this interval is empty, then $\mathcal{J}(\overline{D}) - \mathcal{J}(\overline{D}') = 0$ and the proof is finished. If not, we construct the optimal $\phi^{*'}$. Define $\hat{\theta}(y, z)$ as the type for which the principal is indifferent between giving y or $z > y$, or the unique solution to $J(y, \hat{\theta}(y, z)) = J(z, \hat{\theta}(y, z))$. By proposition 2, the following assignment function is optimal:

$$\phi^{*'}(\theta) = \begin{cases} a & \text{if } \theta \in [\inf \Theta(a, b), \hat{\theta}(a, b)], \\ b & \text{if } \theta \in (\hat{\theta}(a, b), \sup \Theta(a, b)], \\ \phi^*(\theta) & \text{otherwise.} \end{cases} \tag{81}$$

where we observe that $\sup \Theta(a,b) = (\phi^*)^{-1}(b)$. Defining the left generalized inverse as $\phi^\dagger(z) = \sup\{\theta \in \Theta : \phi(\theta) \leq z\}$, we also observe that $\inf \Theta(a,b) = (\phi^*)^\dagger(a)$. Because of this, we have that:

$$\inf \Theta(a,b) = \begin{cases} \min_{x \in \overline{D}: x > a} \hat{\theta}(a,x), \text{if it exists,} \\ \left(\phi^P\right)^{-1}(a), \text{otherwise.} \end{cases} \tag{82}$$

$$\sup \Theta(a,b) = \begin{cases} \max_{x \in \overline{D}: x < b} \hat{\theta}(b,x), \text{if it exists,} \\ \left(\phi^P\right)^{-1}(b), \text{otherwise.} \end{cases} \tag{83}$$

We can now bound the loss in value from the deletion of (a,b) from $\overline{D}$. By the previous arguments, we have that:

$$\begin{aligned} \mathcal{J}(\overline{D}) - \mathcal{J}(\overline{D}') = &\int_{\inf \Theta(a,b)}^{\hat{\theta}(a,b)} \left(J(\phi^*(\theta),\theta) - J(a,\theta)\right) \mathrm{d}F(\theta) \\ &+ \int_{\hat{\theta}(a,b)}^{\sup \Theta(a,b)} \left(J(\phi^*(\theta),\theta) - J(b,\theta)\right) \mathrm{d}F(\theta) \end{aligned} \tag{84}$$

We now proceed in three steps. We first bound the integrands, then bound the limits of integration, and finally put the two together.

Step 1: Bounding the Integrands. We first derive an upper bound for $J(\phi^*(\theta),\theta) - J(x,\theta)$. We expand $J(x,\theta)$ to the second order around $\phi^*(\theta)$. Using Taylor's remainder theorem, and evaluating at $x = \phi^{*'}(\theta)$,

$$J(\phi^{*'}(\theta),\theta) = J(\phi^*(\theta),\theta) + J_x(\phi^*(\theta),\theta)(\phi^{*'}(\theta) - \phi^*(\theta)) + \frac{1}{2} J_{xx}(y(\theta),\theta)(\phi^{*'}(\theta) - \phi^*(\theta))^2 \tag{85}$$

for some $y(\theta) \in [\phi^*(\theta), \phi^{*'}(\theta)] \cup [\phi^{*'}(\theta), \phi^*(\theta)]$. We further apply Taylor's remainder theorem to take a first-order expansion of $J_x(x,\theta)$ around $x = \phi^P(\theta)$ and evaluate at $x = \phi^*(\theta)$:

$$\begin{aligned} J_x(\phi^*(\theta),\theta) &= J_x(\phi^P(\theta),\theta) + J_{xx}(z(\theta),\theta)(\phi^*(\theta) - \phi^P(\theta)) \\ &= J_{xx}(z(\theta),\theta)(\phi^*(\theta) - \phi^P(\theta)) \end{aligned} \tag{86}$$

where the first equality defines the point $z(\theta) \in [\phi^*(\theta), \phi^P(\theta)] \cup [\phi^P(\theta), \phi^*(\theta)]$ and the second uses the fact that $J_x(\phi^P(\theta),\theta) = 0$ by definition, since ϕ^P maximizes J and J is strictly quasiconcave in its first argument. Combining these expansions, we have that:

$$\begin{aligned}|J(\phi^{*'}(\theta),\theta)-J(\phi^*(\theta),\theta)| &\leq |J_x(\phi^*(\theta),\theta)||\phi^{*'}(\theta)-\phi^*(\theta)|+\frac{1}{2}|J_{xx}(y(\theta),\theta)|(\phi^{*'}(\theta)-\phi^*(\theta))^2\\ &\leq |J_{xx}(z(\theta),\theta)|(\phi^{*'}(\theta)-\phi^*(\theta))^2+\frac{1}{2}|J_{xx}(y(\theta),\theta)|(\phi^{*'}(\theta)-\phi^*(\theta))^2\\ &\leq \frac{3}{2}\bar{J}_{xx}(\phi^{*'}(\theta)-\phi^*(\theta))^2\end{aligned} \tag{87}$$

where the second line follows from combining Equation 86 with the observation that $|\phi^*(\theta)-\phi^P(\theta)|\leq|\phi^*(\theta)-\phi^{*'}(\theta)|$. Thus, defining $c=\phi^*(\hat{\theta}(a,b))$, the integrand in the first line of Equation 84 is bounded above by $\frac{3}{2}\bar{J}_{xx}(c-a)^2$ and the integrand in the second line of Equation 84 is bounded above by $\frac{3}{2}\bar{J}_{xx}(b-c)^2$.

Step 2: Bounding the Limits of Integration. We first derive bounds for the limits of integration. There are two approaches to this that we use. The first approach yields Equation 36 and Equation 37. The second approach yields Equation 38.

In the first approach, we observe that $\hat{\theta}(a,b)-\inf\Theta(a,b),\sup\Theta(a,b)-\hat{\theta}(a,b)\leq\sup\Theta(a,b)-\inf\Theta(a,b)\leq\left(\phi^P\right)^{-1}(b)-\left(\phi^P\right)^{-1}(a)$. Both ϕ^P and $(\phi^P)^{-1}$ are monotone and differentiable functions under our maintained assumption that J is twice continuously differentiable and strictly supermodular in (x,θ). In this case, the slope of the inverse function is $((\phi^P)^{-1})'(x)=\frac{1}{(\phi^P)'((\phi^P)^{-1}(x))}$. Moreover, by the implicit function theorem, $(\phi^P)'(\theta)=\frac{J_{x\theta}(\phi^P(\theta),\theta)}{-J_{xx}(\phi^P(\theta),\theta)}$, where by the fact that $\phi^P(\theta)\in(0,\overline{x})$ we must have $-J_{xx}(\phi^P(\theta),\theta)>0$.[24] Therefore, we can write the bound

$$((\phi^P)^{-1})'(x)=\frac{-J_{xx}(x,(\phi^P)^{-1}(x))}{J_{x\theta}(x,(\phi^P)^{-1}(x))}\leq\frac{\sup_{y\in X,\theta\in\Theta}|J_{xx}(y,\theta)|}{\inf_{y\in X,\theta\in\Theta}J_{x\theta}(y,\theta)}=\frac{\bar{J}_{xx}}{\underline{J}_{x\theta}}<\infty \tag{88}$$

where penultimate inequality uses the definitions of $\bar{J}_{xx}$ and $\underline{J}_{x\theta}$; and the last inequality follows from the fact that J twice continuously differentiable and strictly supermodular over the compact set $X\times\Theta$. Thus, we have that:

$$\sup\Theta(a,b)-\inf\Theta(a,b)\leq\frac{\bar{J}_{xx}}{\underline{J}_{x\theta}}(b-a) \tag{89}$$

In the second approach, we suppose that $a<c<b$ are three sequential points in $\overline{D}$, *i.e*, c is isolated, and a and b are the closest elements to c in $\overline{D}$. In this case $\inf\Theta(a,b)=\hat{\theta}(a,c)$ and $\sup\Theta(a,b)=\hat{\theta}(c,b)$. We first bound $\hat{\theta}(a,b)-\hat{\theta}(a,c)$.

To do this, we define $\hat{\theta}(u)=\hat{\theta}(a,c+u)$ and note that $\hat{\theta}(b-c)=\hat{\theta}(a,b)$ and $\hat{\theta}(0)=$

[24] This is vacuously true when J is linear in x so that $J_{xx}=0$ because, in this case, we have $\phi^P(\theta)\in\{0,\overline{x}\}$ for all θ.

$\hat{\theta}(a,c)$. Under this reformulation, the definition of $\hat{\theta}(u)$ can be re-written as $J(c+u,\hat{\theta}(u)) = J(a,\hat{\theta}(u))$. We now implicitly differentiate this to obtain

$$\hat{\theta}'(u) = \frac{-J_x(c+u,\hat{\theta}(u))}{J_\theta(c+u,\hat{\theta}(u)) - J_\theta(a,\hat{\theta}(u))} \tag{90}$$

We now apply Taylor's remainder theorem to $\hat{\theta}(u)$ around $u=0$, evaluated at $u = b-c$, to obtain

$$\hat{\theta}(b-c) = \hat{\theta}(0) + \hat{\theta}'(\tilde{u})(b-c) \tag{91}$$

for some $\tilde{u} \in [0, b-c]$. Using our definitions, this implies

$$\hat{\theta}(a,b) - \hat{\theta}(a,c) = \hat{\theta}(b-c) - \hat{\theta}(0) = \frac{-J_x(c+\tilde{u},\hat{\theta}(\tilde{u}))}{J_\theta(c+\tilde{u}t,\hat{\theta}(\tilde{u})) - J_\theta(a,\hat{\theta}(\tilde{u}))}(b-c) \tag{92}$$

We now bound the numerator and denominator of the first fraction. For the numerator, we apply Taylor's remainder theorem to $J_x(\cdot,\hat{\theta}(\tilde{u}))$ around $x = \phi^P(\hat{\theta}(\tilde{u}))$ to write

$$\begin{aligned} J_x(c+\tilde{u},\hat{\theta}(\tilde{u})) =& J_x(\phi^P(\hat{\theta}(\tilde{u})),\hat{\theta}(\tilde{u})) + J_{xx}(z,\hat{\theta}(\tilde{u}))(c+\tilde{u}-\phi^P(\hat{\theta}(\tilde{u}))) \\ =& J_{xx}(z,\hat{\theta}(\tilde{u}))(c+\tilde{u}-\phi^P(\hat{\theta}(\tilde{u}))) \end{aligned} \tag{93}$$

for some $z \in [c+\tilde{u}, \phi^P(\hat{\theta}(\tilde{u}))]$, where we use $J_x(\phi^P(\theta),\theta) = 0$ in the second line. Moreover, we have that $(c+\tilde{u}-\phi^P(\hat{\theta}(\tilde{u}))) \leq b-a$. Therefore, we have that $|J_x(c+\tilde{u},\hat{\theta}\tilde{u})| < \bar{J}_{xx}(b-a)$. For the denominator, we apply Taylor's remainder theorem to $J_\theta(\cdot,\hat{\theta}(\tilde{u}))$ around $x=a$ to write

$$J_\theta(c+\tilde{u},\hat{\theta}(\tilde{u})) - J_\theta(a,\hat{\theta}(\tilde{u})) = J_{x\theta}(z,\hat{\theta}(\tilde{u}))(c+\tilde{u}-a) \tag{94}$$

for some $z \in [a, c+\tilde{u}]$. We observe that $c+\tilde{u}-a \geq c-a$. Therefore, $|J_\theta(c+\tilde{u},\hat{\theta}(\tilde{u})) - J_\theta(a,\hat{\theta}(\tilde{u}))| \geq \underline{J}_{x\theta}(c-a)$. Combining these two bounds, we deduce that:

$$\hat{\theta}(a,b) - \hat{\theta}(a,c) \leq \frac{\bar{J}_{xx}(b-a)}{\underline{J}_{x\theta}(c-a)}(b-c) \tag{95}$$

To bound, $\hat{\theta}(c,b) - \hat{\theta}(a,b)$ we can apply analogous arguments. By doing this, we obtain:

$$\hat{\theta}(a,b) - \hat{\theta}(c,b) \leq \frac{\bar{J}_{xx}(b-a)}{\underline{J}_{x\theta}(b-c)}(c-a) \tag{96}$$

Step 3: Bounding the Value. Combining steps 1 and 2. We can now derive the payoff bound of Equation 37:

$$
\begin{aligned}
\mathcal{J}(\overline{D}) - \mathcal{J}(\overline{D}') &= \int_{\inf \Theta(a,b)}^{\hat{\theta}(a,b)} \left(J(\phi^*(\theta), \theta) - J(a, \theta)\right) \mathrm{d}F(\theta) \\
&\quad + \int_{\hat{\theta}(a,b)}^{\sup \Theta(a,b)} \left(J(\phi^*(\theta), \theta) - J(b, \theta)\right) \mathrm{d}F(\theta) \\
&\leq \int_{\inf \Theta(a,b)}^{\hat{\theta}(a,b)} \frac{3}{2} \bar{J}_{xx} (c-a)^2 \, \mathrm{d}F(\theta) + \int_{\hat{\theta}(a,b)}^{\sup \Theta(a,b)} \frac{3}{2} \bar{J}_{xx} (b-c)^2 \, \mathrm{d}F(\theta) \\
&\leq \frac{3}{2} \bar{J}_{xx} [(c-a)^2 + (b-c)^2] \int_{\inf \Theta(a,b)}^{\sup \Theta(a,b)} \mathrm{d}F(\theta) \\
&\leq \frac{3}{2} \bar{J}_{xx} [(c-a)^2 + (b-c)^2] \frac{\bar{J}_{xx}}{\underline{J}_{x\theta}} (b-a) \bar{f} \\
&= \frac{3}{2} \frac{\bar{J}_{xx}^2 \bar{f}}{\underline{J}_{x\theta}} (b-a) [(c-a)^2 + (b-c)^2]
\end{aligned} \tag{97}
$$

Observing that $(c-a)^2 + (b-c)^2 \leq (b-a)^2$, we also obtain Equation 36.

Finally, we obtain Equation 38 by combining step 1 with the second approach to step 2. Doing this, we obtain:

$$
\begin{aligned}
\mathcal{J}(\overline{D}) - \mathcal{J}(\overline{D}') &= \int_{\hat{\theta}(a,c)}^{\hat{\theta}(a,b)} \left(J(\phi^*(\theta), \theta) - J(a, \theta)\right) \mathrm{d}F(\theta) \\
&\quad + \int_{\hat{\theta}(a,b)}^{\sup \Theta(c,b)} \left(J(\phi^*(\theta), \theta) - J(b, \theta)\right) \mathrm{d}F(\theta) \\
&\leq \int_{\hat{\theta}(a,c)}^{\hat{\theta}(a,b)} \frac{3}{2} \bar{J}_{xx} (c-a)^2 \, \mathrm{d}F(\theta) + \int_{\hat{\theta}(a,b)}^{\hat{\theta}(c,b)} \frac{3}{2} \bar{J}_{xx} (b-c)^2 \, \mathrm{d}F(\theta) \\
&\leq 3 \frac{\bar{J}_{xx}^2 \bar{f}}{\underline{J}_{x\theta}} (b-a)(c-a)(b-c)
\end{aligned} \tag{98}
$$

Completing the proof.

A.6 Proof of Lemma 3

There are only two $\underline{\delta}$ functions satisfying the properties of Proposition 1 and such that $\underline{\delta}(X) \subseteq \{0, \overline{x}\}$: (i) $\underline{\delta}(x) = 0$ and (ii) $\underline{\delta}(x) = \overline{x}\mathbb{I}[x = \overline{x}]$. Suppose that $\underline{\delta}$ is given by neither (i) or (ii). That is, there exists an $x_0 \in (0, \overline{x}]$ such that $\underline{\delta}'(x_0) \in (0, \overline{x})$. Consider replacing this $\underline{\delta}'$ with $\underline{\delta} = 0$. As $\overline{D}$ is the same under $\underline{\delta}'$ and $\underline{\delta}$, this replacement is strictly better if and only if $\Gamma(\underline{\delta}', \overline{\delta}) > \Gamma(0, \overline{\delta})$ for all $\overline{\delta}$ satisfying the properties of Proposition 1. Strong monotonicity of Γ implies that:

$$\Gamma(\underline{\delta}', \overline{\delta}) - \Gamma(0, \overline{\delta}) \geq \varepsilon \int_X \underline{\delta}'(x)\,\mathrm{d}x > 0 \tag{99}$$

where the area below $\underline{\delta}'$ is greater than zero by monotonicity of $\underline{\delta}'$ and the fact that $\underline{\delta}'(x_0) \in (0, \overline{x}]$ with $x_0 \in (0, \overline{x})$.

A.7 Proof of Lemma 4

Fix $\overline{D} \in \overline{\mathcal{D}}$ and the corresponding $\overline{\delta}$ and fix a sequence $\{a_m, x_m, b_m\}_{m=1}^{\infty} \subseteq \overline{D}$ such that $x_m \in (a_m, b_m)$ and $\overline{D} \cap (a_m, b_m) \to \{x\}$. By strong monotonicity, we have that:

$$\Gamma(\overline{D}) - \Gamma(\overline{D} \setminus (a_m, b_m)) \geq \varepsilon \int_{a_m}^{b_m} \left(\overline{\delta}(z) - b_m\right) \mathrm{d}z = \varepsilon \int_{a_m}^{b_m} \overline{\delta}(z)\,\mathrm{d}z - \varepsilon b_m(b_m - a_m) \tag{100}$$

The cost therefore satisfies Equation 40 if:

$$\begin{aligned} \int_{a_m}^{b_m} \overline{\delta}(z)\,\mathrm{d}z &\leq b_m(b_m - a_m) - (x_m - a_m)(b_m - x_m) \\ &= b_m(b_m - x_m) + x_m(x_m - a_m) \\ &= \int_{a_m}^{b_m} \overline{\delta}_m(z)\,\mathrm{d}z \end{aligned} \tag{101}$$

where $\overline{\delta}_m : [a_m, b_m] \to [0, 1]$ is given by:

$$\overline{\delta}_m(z) = \begin{cases} x_m, & z \in [a_m, x_m], \\ b_m, & z \in (x_m, b_m]. \end{cases} \tag{102}$$

As $a_m, x_m, b_m \in \overline{D}$, observe that $\overline{\delta}_{\overline{D}}(z) \leq \overline{\delta}_m(z)$ for all $z \in [a_m, b_m]$, completing the proof.

A.8 Proof of Lemma 5

Fix a $\overline{D} \in \mathcal{D}$ that is infinite, fix an accumulation point $x \in \overline{D}$ (which must exist by compactness of $\overline{D}$), and fix a number $t > 0$. Define the closed ball of points within a radius t of x as $\overline{B}_t(x)$. Consider the closed set $\overline{B}_t(x) \cap \overline{D}$. There are four mutually exclusive and exhaustive possibilities for this set:

1. $\overline{B}_t(x) \cap \overline{D}$ is a perfect set
   (a) Moreover, $\overline{B}_t(x) \cap \overline{D}$ is somewhere dense in $\overline{B}_t(x)$. In this case, there exists an open interval $(a, b) \subseteq \overline{B}_t(x) \cap \overline{D}$. Claim I below shows that $\overline{D} \setminus (a, b)$ is strictly

preferred to $\overline{D}$, implying the suboptimality of $\overline{D}$.

(b) Moreover, $\overline{B}_t(x) \cap \overline{D}$ is nowhere dense in $\overline{B}_t(x)$. In case 2(a), $\overline{B}_t(x) \cap \overline{D}$ is a compact, nowhere dense, perfect set. The arguments made in case 2(a) below establish the existence of an accumulation point $w \in \overline{B}_t(x) \cap \overline{D}$ that is isolated from the left: there exists $\eta > 0$ such that $(w - \eta, w)$ has empty intersection with $\overline{B}_t(x) \cap \overline{D}$.[25] Given such a point, Claim II below constructs a variation $\overline{D} \setminus (a, b)$, where $w \in (a, b)$ that is strictly preferred to $\overline{D}$.

2. $\overline{B}_t(x) \cap \overline{D}$ is not a perfect set

   (a) Moreover, $\overline{B}_t(x) \cap \overline{D}$ is uncountably infinite. We define $A = \overline{B}_t(x) \cap \overline{D}$. In this case, A is not a perfect set, is uncountably infinite, and contains no intervals. As A is closed, by the Cantor-Bendixson theorem (see, *e.g.,* p. 67 of Apostol, 1974) it can be decomposed as $A = A^P \cup A^C$, where $A^P \cap A^C = \emptyset$, A^P is a perfect set, and A^C is at most countably infinite. As A contains no intervals, it is nowhere dense in $\overline{B}_t(x)$. Thus, A^P is nowhere dense in $\overline{B}_t(x)$. Therefore, there exists an open set $U \subset \overline{B}_t(x)$ such that $U \cap A^P = \emptyset$. Fix an arbitrary point $z \in U$ and define $w = \min\{x' \in A^P : x' \geq z\}$. These points are well defined since A^P is a perfect, hence closed, set. Moreover, we have that $w > z$. Now consider the set $Z = [z, w) \cap A$. Observe that $Z = [z, w) \cap A^C$. Toward a contradiction, assume that there exists $w' \in [z, w) \cap A^P$. Since $z \leq w' < w$ and $w' \in A^P$, it follows that $\min\{x' \in A^P : x' \geq z\} \leq w' < w$, yielding a contradiction to the definition of w. There are three possibilites for Z: it is empty, it is a finite set, or it is a countably infinite set. If Z is an empty set, then $(z, w) \cap A = \emptyset$. If Z is a finite set, define $z' = \arg\min_{x' \in Z} w - x'$. In this case, we have that $(z', w) \cap A = \emptyset$. In both of these cases, we have shown that there exists $w \in A$ that is isolated from the left and is an accumulation point in A. Claim II below uses this fact to construct a variation $\overline{D} \setminus (a, b)$ that is strictly preferred to $\overline{D}$. If Z is a countably infinite set, then fix an accumulation point $y \in Z$ and consider the set $\overline{B}_{t'}(y) \cap \overline{D}$ for some $t' > 0$ such that $\overline{B}_{t'}(y) \subset Z$. In this case, $\overline{B}_{t'}(y) \cap \overline{D}$ is a countably infinite set and we have shown that $y \in \overline{D}$ is an accumulation point of $\overline{D}$. Claim III below uses this fact to construct a variation $\overline{D} \setminus (a, b)$ that is strictly preferred to $\overline{D}$.

   (b) Moreover, $\overline{B}_t(x) \cap \overline{D}$ is countably infinite. Claim III constructs a variation $\overline{D} \setminus (a, b)$ that is strictly preferred to $\overline{D}$.

We now prove the three claims in turn, thus completing the proof.

[25] This follows from the construction of case 2(a). In particular, by observing that $A^C = \emptyset$ (the set is already perfect in this case), the same arguments as in case 2(a) yield $Z = \emptyset$, implying the existence of an isolated point from the left in $\overline{B}_t(x) \cap \overline{D}$.

Claim I: "Intervals". Suppose that $\overline{D}$ contains an interval I. Let x be the midpoint of such an interval and consider a sequence of points $a_m = x - \frac{t}{m}$, $x_m = x$, and $b_m = x + \frac{t}{m}$, where $t > 0$ is small enough such that $(x - t, x + t)$ is contained in I. We use Equation 36 from Lemma 2. In particular, for every m, we have that:

$$\mathcal{J}(\overline{D}) - \mathcal{J}\left(\overline{D} \setminus \left(x - \frac{t}{m}, x + \frac{t}{m}\right)\right) \leq 12 \frac{\bar{J}_{xx}^2 \bar{f}}{\underline{J}_{x\theta}} t^3 m^{-3} \tag{103}$$

We observe that $\overline{D} \cap (x - \frac{t}{m}, x + \frac{t}{m}) = (x - \frac{t}{m}, x + \frac{t}{m})$ for all m by construction. Moreover, the topological limit of $(x - \frac{t}{m}, x + \frac{t}{m})$ is $\{x\}$. Thus, by Lemma 4, there exists M such that for all $m \geq M$, we have that:

$$\Gamma(\overline{D}) - \Gamma\left(\overline{D} \setminus \left(x - \frac{t}{m}, x + \frac{t}{m}\right)\right) \geq \varepsilon t^2 m^{-2} \tag{104}$$

Thus, for all $m > \max\left\{M, 12 \frac{\bar{J}_{xx}^2 \bar{f}}{\underline{J}_{x\theta}} \frac{t}{\varepsilon}\right\}$ we have that:

$$\mathcal{J}(\overline{D}) - \Gamma(\overline{D}) < \mathcal{J}\left(\overline{D} \setminus \left(x - \frac{t}{m}, x + \frac{t}{m}\right)\right) - \Gamma\left(\overline{D} \setminus \left(x - \frac{t}{m}, x + \frac{t}{m}\right)\right) \tag{105}$$

which contradicts the optimality of $\overline{D}$.

Claim II: "Perfect and Nowhere Dense Sets". Suppose that an accumulation point $x^* \in \overline{D}$ is isolated from the left. That is, there exists $\eta > 0$ such that $(x^* - \eta, x^*) \cap \overline{D} = \emptyset$. Define also the point $y = \max\{z \in [0, x^* - \eta] \cap \overline{D}\}$, which is well-defined by compactness of $\overline{D}$. Next, consider the constant sequence $a_m = y$ and the sequence $\{b_m\}$ equal to a monotone decreasing sequence of points in $\overline{D}$ that converges to x^*. Because of the Bolzano-Weierstrass theorem, such a sequence always exists as x^* is a limit point. Thus, we have from statement 2 of Lemma 2 (Equation 37) that there exists a sequence of points $z_m \in (x^*, b_m) \cap \overline{D}$ such that:

$$\begin{aligned} &\mathcal{J}(\overline{D}) - \mathcal{J}(\overline{D} \setminus (y, b_m)) = \mathcal{J}(\overline{D}) - \mathcal{J}(\overline{D} \setminus [x^*, b_m)) \\ &\leq \frac{3}{2} \frac{\bar{J}_{xx}^2 \bar{f}}{\underline{J}_{x\theta}} (b_m - x^*) \left[(b_m - z_m)^2 + (z_m - x^*)^2\right] \leq \frac{3}{2} \frac{\bar{J}_{xx}^2 \bar{f}}{\underline{J}_{x\theta}} (b_m - y) \left[(b_m - x^*)^2\right] \end{aligned} \tag{106}$$

We now fix the sequence $x_m = x^*$ and observe that the topological limit of $(y, b_m) \cap \overline{D}$ is $\{x^*\}$. Thus, by Lemma 4, we have that there exists M such that for all $m \geq M$, we have that:

$$\Gamma(\overline{D}) - \Gamma(\overline{D} \setminus (y, b_m)) \geq \varepsilon (x^* - y)(b_m - x^*) \tag{107}$$

As $b_m - x^*$ is common to both terms we have that for all $m \geq M$ that:

$$\begin{aligned}
&\Gamma(\overline{D}) - \Gamma(\overline{D} \setminus (y, b_m)) - (\mathcal{J}(\overline{D}) - \mathcal{J}(\overline{D} \setminus (y, b_m))) \\
&\geq (b_m - x^*) \left[\varepsilon(x^* - y) - \frac{3}{2}\frac{\bar{J}_{xx}^2 \bar{f}}{\underline{J}_{x\theta}}(b_m - x)(b_m - y)\right]
\end{aligned} \tag{108}$$

As $b_m \to x^*$, we have that there exists a $\hat{M}$ such that $\left[\varepsilon(x^* - y) - \frac{3}{2}\frac{\bar{J}_{xx}^2 \bar{f}}{\underline{J}_{x\theta}}(b_m - x)(b_m - y)\right] > 0$ for all $m \geq \hat{M}$, which implies that for all $m \geq \max\{M, \hat{M}\}$:

$$\mathcal{J}(\overline{D}) - \Gamma(\overline{D}) < \mathcal{J}(\overline{D} \setminus (y, b_m)) - \Gamma(\overline{D} \setminus (y, b_m)) \tag{109}$$

This contradicts the optimality of $\overline{D}$.

Claim III: "Countably Infinite Sets". Suppose that x is an accumulation point of $\overline{D}$ and $\overline{B}_t(x) \cap \overline{D}$ is a countably infinite set for some $t > 0$. As $\overline{D}$ is countably infinite, we know that every neighborhood of x contains an isolated point. Let $\{x_m\} \subset \overline{B}_t(x) \cap \overline{D}$ be a monotone sequence of isolated points such that $x_m \to x$. As x_m is isolated, we may define $a_m = \max\{y \in \overline{D} : y < x_m\}$ and $b_m = \min\{y \in \overline{D} : y > x_m\}$. By statement 3. in Lemma 2 (Equation 38), we have that:

$$\mathcal{J}(\overline{D}) - \mathcal{J}(\overline{D} \setminus \{x_m\}) \leq 3\frac{\bar{J}_{xx}^2 \bar{f}}{\underline{J}_{x\theta}}(b_m - a_m)(x_m - a_m)(b_m - x_m) \tag{110}$$

By construction, we have that $x_m \in (a_m, b_m)$. Moreover, $\overline{D} \cap (a_m, b_m) = \{x_m\}$, the topological limit of which is $\{x\}$ as $x_m \to x$. Thus, by Lemma 4, we have that there exists M such that for all $m \geq M$, we have that:

$$\Gamma(\overline{D}) - \Gamma(\overline{D} \setminus \{x_m\}) \geq \varepsilon(x_m - a_m)(b_m - x_m) \tag{111}$$

Factoring $(x_m - a_m)(b_m - x_m)$ from both expressions, we have that:

$$\begin{aligned}
&\Gamma(\overline{D}) - \Gamma(\overline{D} \setminus \{x_m\}) - (\mathcal{J}(\overline{D}) - \mathcal{J}(\overline{D} \setminus \{x_m\})) \\
&\geq (x_m - a_m)(b_m - x_m) \left[\varepsilon - 3\frac{\bar{J}_{xx}^2 \bar{f}}{\underline{J}_{x\theta}}(b_m - a_m)\right]
\end{aligned} \tag{112}$$

As $a_m, b_m \to x$, we have that there exists $\hat{M}$ such that $\varepsilon - 3\frac{\bar{J}_{xx}^2 \bar{f}}{\underline{J}_{x\theta}}(b_m - a_m) > 0$ for all $m \geq \hat{M}$. This implies that for all $m \geq \max\{M, \hat{M}\}$ that:

$$\mathcal{J}(\overline{D}) - \Gamma(\overline{D}) < \mathcal{J}(\overline{D} \setminus \{x_m\}) - \Gamma(\overline{D} \setminus \{x_m\}) \tag{113}$$

which contradicts the optimality of $\overline{D}$.

A.9 Proof of Lemma 6

An optimal $\overline{D}^*$ exists by Lemma 1 and must be finite by Lemma 5. Furthermore, given that any optimal $\overline{D}^*$ is finite, we can express any such set as a finite set of ordered points $\{x_1, \dots, x_{K^*}\}$ for some $K^* \in \mathbb{N}$. Take any three sequential points $x_{m-1}, x_m, x_{m+1} \in \overline{D}^*$. We can apply statement 3 of Lemma 2 (Equation 38) to bound the loss from eliminating contractibility at x_m:

$$\mathcal{J}(\overline{D}^*) - \mathcal{J}(\overline{D}^* \setminus \{x_m\}) \leq 3\frac{\bar{J}_{xx}^2 \bar{f}}{\underline{J}_{x\theta}}(x_m - x_{m-1})(x_{m+1} - x_m)(x_{m+1} - x_{m-1}) \tag{114}$$

Moreover, we can take constant sequences $a_n = x_{m-1}$, $\tilde{x}_n = x_m$ $b_n = x_{m+1}$ for all $n \in \mathbb{N}$. $a_n, \tilde{x}_n, b_n \in \overline{D}^*$ for all $n \in \mathbb{N}$ and $\overline{D}^* \cap (a_n, b_n) = \{x_m\}$ for all $n \in \mathbb{N}$. Thus, by strong monotonicity of Γ, Lemma 4 implies that:

$$\Gamma(\overline{D}^*) - \Gamma(\overline{D}^* \setminus \{x_m\}) \geq \varepsilon(x_m - x_{m-1})(x_{m+1} - x_m) \tag{115}$$

Optimality of $\overline{D}^*$ requires that $\mathcal{J}(\overline{D}^*) - \mathcal{J}(\overline{D}^* \setminus \{x_m\}) \geq \Gamma(\overline{D}^*) - \Gamma(\overline{D}^* \setminus \{x_m\})$. Combining this with Inequalities 114 and 115, we have that:

$$3\frac{\bar{J}_{xx}^2 \bar{f}}{\underline{J}_{x\theta}}(x_m - x_{m-1})(x_{m+1} - x_m)(x_{m+1} - x_{m-1}) \geq \varepsilon(x_m - x_{m-1})(x_{m+1} - x_m) \tag{116}$$

Dividing both sides by $(x_{m+1} - x_m)(x_m - x_{m-1})$ yields

$$x_{m+1} - x_{m-1} \geq \frac{\varepsilon}{3}\frac{\underline{J}_{x\theta}}{\bar{J}_{xx}^2 \bar{f}} \tag{117}$$

Thus, we have that:

$$\bar{x} \geq x_{K^*} - x_1 = \sum_{j=1}^{\lfloor K^*/2 \rfloor} x_{2j+1} - x_{2j-1} \geq K^* \frac{\varepsilon}{6}\frac{\underline{J}_{x\theta}}{\bar{J}_{xx}^2 \bar{f}} \tag{118}$$

Re-arranging this equation yields the desired bound.

A.10 Proof of Proposition 4

We first derive the optimal allocation. As J is strictly single-crossing, $J(x_k, \theta) - J(x_{k-1}, \theta) = 0$ has no solution if and only if (i) $J(x_k, 0) - J(x_{k-1}, 0) > 0$ and (ii) $J(x_k, 1) - J(x_{k-1}, 1) < 0$.

As J is strictly quasi-concave, if $J(x_k, 0) - J(x_{k-1}, 0) > 0$, then $J(\cdot, 0)$ is strictly increasing at x_{k-1}, and therefore at all x_j for $j \leq k-1$. Thus, if $J(x_k, 0) - J(x_{k-1}, 0) > 0$ holds for k, it holds for all $j \leq k$. Define $\underline{k} = \max\{k \in \{1, \ldots, K\} : J(x_k, 0) - J(x_{k-1}, 0) > 0\}$, with the convention that $\underline{k} = 1$ if this set is empty. Similarly, if $J(x_k, 1) - J(x_{k-1}, 1) < 0$, then $J(\cdot, 1)$ is strictly decreasing at x_k. Thus, if $J(x_k, 1) - J(x_{k-1}, 1) < 0$ holds for k, it holds for all $j \geq k$. Define $\overline{k} = \min\{k \in \{1, \ldots, K\} : J(x_k, 1) - J(x_{k-1}, 1) < 0\}$, with the convention that $\overline{k} = K$ if this set is empty. As J is strictly single crossing, $\overline{k} > \underline{k}$. We now have that $J(x_k, \theta) - J(x_{k-1}, \theta) = 0$ has a solution if and only if $k \in \{\underline{k}+1, \ldots, \overline{k}-1\}$ (if $\overline{k} = \underline{k}+1$, then this set is empty). For all $k \geq \overline{k}$, we have that $\hat{\theta}_k = 1$. For all $k \leq \underline{k}$, we have that $\hat{\theta}_k = 0$. For all $k \in \{\underline{k}+1, \ldots, \overline{k}-1\}$, we have that $\hat{\theta}_k$ is the unique solution to $J(x_k, \hat{\theta}_k) = J(x_{k-1}, \hat{\theta}_k)$. As J is strictly quasi-concave, we know that $\phi^P(\hat{\theta}_k) \in (x_{k-1}, x_k)$, which implies that $\underline{\phi}(\hat{\theta}_k) = x_{k-1}$ and $\overline{\phi}(\hat{\theta}_k) = x_k$. Thus, by Proposition 2, we have that $\phi^*(\theta) = x_k$ for all $\theta \in (\hat{\theta}_k, \hat{\theta}_{k+1}]$.

We now derive the tariff that supports this allocation. Applying Equation 12 from Lemma 8, we have that:

$$
\begin{aligned}
T(x_k) &= u(x_k, \hat{\theta}_k) - \mathbb{I}[k \geq 2] \sum_{j=1}^{k-1} \int_{\hat{\theta}_j}^{\hat{\theta}_{j+1}} u_\theta(x_j, s)\, \mathrm{d}s \\
&= u(x_k, \hat{\theta}_k) - \mathbb{I}[k \geq 2] \sum_{j=1}^{k-1} \left[u(x_j, \hat{\theta}_{j+1}) - u(x_j, \hat{\theta}_j) \right] \qquad (119) \\
&= u(x_1, 0) + \mathbb{I}[k \geq 2] \sum_{j=2}^{k} \left[u(x_j, \hat{\theta}_j) - u(x_{j-1}, \hat{\theta}_j) \right]
\end{aligned}
$$

where the second equality computes the integrals and the final equality telescopes the summation. Observing that $x_1 = 0$ and $u(0, 0) = 0$ completes the proof.

A.11 Proof of Proposition 5

Observe for any uncertain cost of distinguishing indexed by a compactly supported $\mu \in \Delta(\mathcal{G})$ and $\lambda \in (-\infty, \infty)$ that we can write:

$$
\Gamma_0^{\mu,\lambda}(\overline{\delta}_{\mathbf{x}}) = \frac{1}{\lambda} \log \left(\int_{\mathcal{G}} \exp \left(\lambda \sum_{k=1}^{K} \int_{x_{k-1}}^{x_k} [G(\overline{x}, y) - G(x_k, y)]\, \mathrm{d}y \right) \mathrm{d}\mu(g) \right) \qquad (120)
$$

For every $k \in \{2, ..., K\}$, define $I(x_{k-1}, x_k) = \int_{x_{k-1}}^{x_k} [G(\overline{x}, y) - G(x_k, y)]\, \mathrm{d}y$. By Leibniz's rule, for every $k \in \{2, ..., K-1\}$, this is continuously differentiable in (x_{k-1}, x_k) when $x_{k-1} < x_k$, and $I(x_{K-1}, x_K)$ is continuously differentiable in x_{K-1} when $x_{K-1} < \overline{x}$. This implies that

$\Gamma_0^g(\overline{\delta}_{\mathbf{x}})$ is continuously differentiable in $0 = x_1 < \ldots < x_K = \overline{x}$ for all $g \in \mathcal{G}$. Finally, because $\lambda \in (-\infty, \infty)$ and μ has compact support, the chain rule and the dominated convergence theorem imply that $\Gamma_0^{\mu,\lambda}(\overline{\delta}_{\mathbf{x}})$ is continuously differentiable in $0 = x_1 < \ldots < x_K = \overline{x}$.

A.12 Proof of Proposition 6

We first introduce some preliminary notation. Given a vector $\mathbf{x} = (x_1, x_2, ..., x_{K-1}, x_K) \in \mathbb{R}^K$ such that $x_1 = 0$, $x_K = \overline{x}$, and $K \geq 2$, for all $\varepsilon > 0$, we let $(x_k + \varepsilon, x_{-k}) \in \mathbb{R}^K$ denote the vector where we replace x_k with $x_k + \varepsilon$ for some $k \in \{2, ..., K-1\}$.

Because Γ is strongly monotone, Theorem 1 implies that any optimal set of self-enforcing recommendations is finite with cardinality less than B. Fix one such optimal set $\overline{D}^* = \{x_1^*, ..., x_{K^*}^*\}$ where $K^* \leq B$ is its cardinality and where $0 = x_1^* < \ldots < x_{K^*}^* = \overline{x}$. Let $\mathbf{x}^* \in X^{K^*}$ denote the totally order vector stacking the elements of $\overline{D}^*$. Because the latter is optimal, $\mathbf{x}^*$ must solve Problem 22. In particular, because Γ is finitely differentiable, $\mathbf{x}^*$ must satisfy the first-order condition

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{J}(\overline{\delta}_{(x_k^*+\varepsilon, x_{-k}^*)})|_{\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Gamma(\overline{\delta}_{(x_k^*+\varepsilon, x_{-k}^*)})|_{\varepsilon=0} \quad \forall k \in \{2, \ldots, K^*-1\} \tag{121}$$

By Proposition 4, for ε small enough as $x_k \in (x_{k-1}, x_{k+1})$, we have that $\mathcal{J}(\overline{\delta}_{(x_k^*+\varepsilon, x_{-k}^*)}) = \int_\Theta J(\phi_\varepsilon^*(\theta), \theta)\mathrm{d}F(\theta)$ where ϕ_ε^* is defined in Equation 21 with $\overline{D}_\varepsilon = \{x_1^*, \ldots, x_k^* + \varepsilon, \ldots, x_{K^*}^*\}$. With this, the left-hand-side of (121) is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{J}(\overline{\delta}_{(x_k^*+\varepsilon, x_{-k}^*)})|_{\varepsilon=0} &= \int_{\hat{\theta}_k}^{\hat{\theta}_{k+1}} J_x(x_k^*, \theta)\,\mathrm{d}F(\theta) + \\ &\quad \frac{\partial}{\partial x_k^*}\hat{\theta}_k\left(J(x_k^*, \hat{\theta}_k) - J(x_{k-1}^*, \hat{\theta}_k)\right) f(\hat{\theta}_k) + \\ &\quad + \frac{\partial}{\partial x_k^*}\hat{\theta}_{k+1}\left(J(x_{k+1}^*, \hat{\theta}_{k+1}) - J(x_k^*, \hat{\theta}_{k+1})\right) f(\hat{\theta}_{k+1}) \\ &= \int_{\hat{\theta}_k}^{\hat{\theta}_{k+1}} J_x(x_k^*, \theta)\,\mathrm{d}F(\theta)\end{aligned} \tag{122}$$

where, in the second equality, we use the fact that $J(x_k^*, \hat{\theta}_k) = J(x_{k-1}^*, \hat{\theta}_k)$ by definition. By construction, the right-hand-side of (121) is $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Gamma(\overline{\delta}_{(x_k^*+\varepsilon, x_{-k}^*)})|_{\varepsilon=0} = \frac{\partial}{\partial x_k}\Gamma(\overline{\delta}_{\mathbf{x}^*})$, hence we obtain Equation 23. Finally, again by definition, we have that $x_1 = 0$ and $x_{K^*} = 1$.

A.13 Proof of Proposition 7

We split the argument into three parts. We first map the problem to one that satisfies the general assumptions of Section 2. We next calculate the optimal contract in the transformed problem. We finally map the allocation and tariff back to the original problem.

Step 1: Transformation of the Problem

We define $x = 1 - e \in X = [0, 1]$ as the agent's shirking and $\theta = 1 - \vartheta \sim F = U[0, 1]$ as their unproductivity. We define transformed preferences for the agent,

$$u(x, \theta) = \tilde{u}(1 - x, 1 - \theta) - \tilde{u}(1, 1 - \theta) = (a\theta + b)x - b\frac{x^2}{2} \tag{123}$$

This describes how much agent θ enjoys shirking at level x relative to providing full effort. The agent's preference u is strict single-crossing in (x, θ), since $u_{x\theta} = a > 0$; increasing in both x and θ on the relevant domain; and satisfies $u(0, \theta) = 0$ for all $\theta \in [0, 1]$. We define the transformed payoff for the principal,

$$\pi(x) = \tilde{\pi}(1 - x) - \tilde{\pi}(1) = -cx \tag{124}$$

This measures the revenue lost by allowing degree x of shirking, relative to receiving full effort. The principal's payoff satisfies $\pi(0) = 0$. We finally observe that costs of distinguishing can be written in terms of shirking when we define the contractibility correspondence $C = 1 - \tilde{C}$. In particular, we observe that

$$\Gamma(C) = \kappa \left(\int_0^1 \underline{\delta}(x)\,\mathrm{d}x + \int_0^1 (1 - \overline{\delta}(x))\,\mathrm{d}x \right) \tag{125}$$

where $\underline{\delta}(x) = \min C(x)$ and $\overline{\delta}(x) = \max C(x)$.

The (e, ϑ) problem satisfies the assumptions of Lemma 9 in the case of monotone *decreasing* preferences. In this case, the result implies that the principal problem given a set of self-enforcing efforts $\underline{D} \subseteq E$ is

$$\begin{aligned} \tilde{\mathcal{J}}(\underline{D}) := \max_{\tilde{\phi}} \quad & \int_0^1 \tilde{J}(e, \vartheta)\,\mathrm{d}\tilde{F}(\vartheta) \\ \text{s.t.} \quad & \tilde{\phi}(\vartheta') \geq \tilde{\phi}(\vartheta), \tilde{\phi}(\vartheta) \in \underline{D}, \quad \vartheta, \vartheta' \in [0, 1] : \vartheta' \geq \vartheta \end{aligned} \tag{126}$$

where the virtual surplus function in the integral is

$$\begin{aligned}\tilde{J}(e,\vartheta) &= \tilde{u}(e,\vartheta)+\tilde{\pi}(e)-\frac{1-\tilde{F}(\vartheta)}{\tilde{f}(\vartheta)}\tilde{u}_{\vartheta}(e,\vartheta)\\ &= -a(1-\vartheta)e-b\frac{e^2}{2}+ce-(1-\vartheta)ae\\ &= -a\theta(1-x)-b\frac{(1-x)^2}{2}+c(1-x)-\theta a(1-x)\\ &= (2a\theta+b-c)x-b\frac{x^2}{2}+\left(c-\frac{b}{2}\right)\end{aligned} \tag{127}$$

Using this substitution, we observe the following mapping to the (x,θ) problem:

$$\begin{aligned}\tilde{\mathcal{J}}(\underline{D})-\left(c-\frac{b}{2}\right)=\mathcal{J}(\overline{D})=\max_{\phi}\quad &\int_0^1\left((2a\theta+b-c)\phi(\theta)-b\frac{\phi(\theta)^2}{2}\right)\mathrm{d}F(\theta)\\ \text{s.t.}\quad &\phi(\theta')\geq\phi(\theta),\phi(\theta)\in\overline{D},\quad \theta,\theta'\in[0,1]:\theta'\geq\theta\end{aligned} \tag{128}$$

where $\overline{D}=\{(1-e)\in X: e\in\underline{D}\}$. Next, for every correspondence $C:X\rightrightarrows X$, define the effort-based correspondence $(1-C):E\rightrightarrows E$ by $(1-C)(\zeta)=\{1-x\in E: x\in C(1-\zeta)\}$ for all $\zeta\in E$. We next map the costs to the case with increasing preferences. We first observe that $\Gamma(C)=\tilde{\Gamma}(1-C)$.

Combining these steps, we obtain that the original problem is equivalent to solving the transformed problem with virtual surplus $J(x,\theta)=(2a\theta+b-c)x-b\frac{x^2}{2}$ to obtain the optimal extent of contractibility.

Theorem 1 implies that any optimal contractibility correspondence can be represented by a coarse set of self-enforcing shirking recommendations, $\overline{D}=\{x_1,\ldots,x_K\}$. In the transformed problem, these map to a set of self-enforcing effort recommendations $\{1-x_1,\ldots,1-x_K\}$. To apply Proposition 6, we must also establish that (without loss of optimality) $x_1=0$ and $x_K=\overline{x}=1$. *Excludability* guarantees that $e_1=0$ (zero effort, or full shirking) is perfectly contractible. An additional simple argument is required to establish also that (without loss of optimality) we can restrict attention to the case in which $e_K=1$ (full effort or zero shirking) is also contractible.[26] Imagine it were not, and optimal contractibility were represented by $\tilde{C}$. Then, construct a variant that perturbs $\tilde{C}$ such that $e=1$ is fully contractible: $\tilde{C}'(\zeta)=\tilde{C}(\zeta)$ for $e\in[0,1)$ and $\tilde{C}'(1)=\{1\}$. We observe that $\tilde{\Gamma}(\tilde{C}')=\tilde{\Gamma}(\tilde{C})$: this can be shown by direct calculation, or by appeal to the L_1-continuity of the induced costs over

[26] This presents one small asymmetry in our analysis of cases with increasing and decreasing preferences: in the problem with increasing preferences, the axioms on contractibility (in particular, the implied *upper* semi-continuity of $\underline{\delta}$) suffice to establish that $\overline{x}\in C(\overline{x})$ and therefore $\overline{x}$ is a self-enforcing recommendation. Here, we need a separate argument that relies on the (maintained) L_1-continuity of costs.

$\underline{\delta}, \overline{\delta}$. But under $\tilde{C}'$ there are weakly more more self-enforcing recommendations. Thus, $\tilde{C}'$ obtains weakly higher value. Hence, it is without loss of optimal to restrict attention to contractibility containing $e = 1$ ($x = 0$) as a self-enforcing recommendation. Together, this argument shows that we can apply Proposition 6 to the transformed problem.

Step 2: Optimal Contract in the Transformed Problem

We leverage our characterization of the optimal contract in Proposition 6 to set up the optimization problem in closed form. The virtual surplus function in this setting is given by Equation 30. Equation 22 gives the principal's interim payoff under the optimal contract conditional on any set of K contractible actions $\{x_k\}_{k=1}^K$.

We first solve for the candidate optimal contract that solves the variational first-order condition in Proposition 6 for each K. This first-order condition for $k \in \{2, K-1\}$ is

$$\int_{\hat{\theta}_k}^{\hat{\theta}_{k+1}} (2a\theta + b(1 - x_k) - c)\, \mathrm{d}\theta - \kappa(-2x_k + x_{k-1} + x_{k+1}) = 0 \tag{129}$$

where $\hat{\theta}_k = \frac{b}{4a}(x_k + x_{k-1}) + \frac{c-b}{2a}$. Calculating the integral and evaluating at $\hat{\theta}_k$, we write

$$\begin{aligned} \kappa(-2x_k + x_{k-1} + x_{k+1}) &= \left[a\theta^2 + (b - c - bx_k)\theta\right]_{\hat{\theta}_k}^{\hat{\theta}_{k+1}} \\ &= (\hat{\theta}_{k+1} - \hat{\theta}_k)\left[a(\hat{\theta}_{k+1} + \hat{\theta}_k) + b - c - bx_k\right] \\ &= \frac{b^2}{16a}(x_{k+1} - x_{k-1})(x_{k+1} + x_{k-1} - 2x_k) \end{aligned} \tag{130}$$

This condition can be re-arranged to obtain

$$(x_{k+1} + x_{k-1} - 2x_k)\left[\frac{b^2}{16a}(x_{k+1} - x_{k-1}) - \kappa\right] = 0 \tag{131}$$

This equation has two solutions,

$$x_k = \frac{x_{k+1} + x_{k-1}}{2}, \qquad\qquad x_{k+1} = x_{k-1} + \Delta \tag{132}$$

where $\Delta = \frac{16a\kappa}{b^2}$. We now separately consider each case.

Case 1: Uniform Grid. From the boundary conditions, we have that $x_1 = 0$ and $x_K = 1$. Thus, we have that:

$$x_k = \frac{x_{k+1} + x_{k-1}}{2} \implies x_k = \frac{k-1}{K-1} \tag{133}$$

We can verify that this is a local maximum by checking the Hessian is negative definite at this solution. We calculate that:

$$\begin{aligned} \frac{\partial^2 \mathcal{J}}{\partial x_k^2} &= H^{\mathcal{J}}_{k-1,k-1} = -\frac{b^2}{4a(K-1)} + 2\kappa =: \Lambda \\ \frac{\partial^2 \mathcal{J}}{\partial x_k \partial x_{k+1}} &= H^{\mathcal{J}}_{k,k-1} = H^{\mathcal{J}}_{k-1,k} = \frac{b^2}{8a(K-1)} - \kappa = -\frac{1}{2}\Lambda \end{aligned} \tag{134}$$

where we note that row and column $k-1$ of $H^{\mathcal{J}}$ corresponds to the variable x_k and in the first equality we define Λ. Thus, the Hessian is a tridiagonal Toeplitz matrix, which implies that the Eigenvalues are, by Theorem 2.2 of Kulkarni, Schmidt, and Tsui (1999), given by:

$$\lambda_k = \Lambda \left(1 + \cos\left(\frac{k-1}{K}\pi\right)\right) \tag{135}$$

for $k \in \{2, \ldots, K-1\}$. As $\cos\left(\frac{k-1}{K}\pi\right) > -1$ for all such k, we have that $\text{sgn}(\lambda_k) = \text{sgn}(\Lambda)$. Thus, the Hessian is negative definite if and only if:

$$K < \bar{K} = 1 + \frac{b^2}{8a\kappa} \tag{136}$$

We will later verify that this holds whenever K is set optimally, confirming the optimality of the uniform grid solution.

Case 2: Alternating Grid. Under the second solution, it must be the case that even points form a uniform grid with spacing $\Delta = \frac{16a\kappa}{b^2}$ and the odd points form a uniform grid with spacing $\Delta = \frac{16a\kappa}{b^2}$. When K is odd, given the boundary conditions that $x_1 = 0$ and $x_K = 1$, we have that this is possible only when $K = 2 + \frac{2}{\Delta}$, which is itself only possible when $\frac{b^2}{8a\kappa}$ is an odd integer. When K is even, the solution must be $x_k = \frac{k-1}{2}\Delta$ for k odd, and $x_k = 1 - \frac{K-k}{2}\Delta$ for k even. This is possible for any even $K < 2 + \frac{2}{\Delta}$.

We next show that the alternating grid is *not* a local maximum of the objective function and therefore can be ignored. For a local maximum, a necessary condition is that the Hessian is negative semidefinite. We will show the existence of a vector $x \in \mathbb{R}^{K-2}$ such that $v \neq 0$ an $v' H^{\mathcal{J}} v > 0$, which implies that $H^{\mathcal{J}}$ is not negative semidefinite. To do this, we first evaluate the second-order conditions at the conjectured alternating grid solution. These simplify to

$$\begin{aligned} \frac{\partial^2 \mathcal{J}}{\partial x_k^2} &= H^{\mathcal{J}}_{k-1,k-1} = -\frac{b^2}{8a}\Delta + 2\kappa = 0 \\ \frac{\partial^2 \mathcal{J}}{\partial x_k \partial x_{k+1}} &= H^{\mathcal{J}}_{k,k-1} = H^{\mathcal{J}}_{k-1,k} = \frac{b^2}{8a}(x_{k+1} - x_k) - \kappa \end{aligned} \tag{137}$$

Using this, we define $v_k = z_{k-1} - z_k$, where z_k denotes the unit vector in dimension k. This direction corresponds to increasing x_k and decreasing x_{k+1}. We calculate

$$v_k' H^{\mathcal{J}} v_k = 2\left(\kappa - \frac{b^2}{8a}(x_{k+1} - x_k)\right) \tag{138}$$

We now split the proof into two cases. First, consider the case in which $K > 4$. In this case, there must exist some x_k, x_{k+1} such that $x_{k+1} - x_k < \frac{\Delta}{2}$, since the grid is not uniform. Then,

$$v_k' H^{\mathcal{J}} v_k > 2\left(\kappa - \frac{\Delta b^2}{16a}\right) > 0 \tag{139}$$

and, as desired, we have shown that the Hessian is not negative definite. Next, we consider the case in which $K = 4$. In this case, we take two candidate vectors. The first is $u = z_1 + z_2$, and we observe

$$u' H^{\mathcal{J}} u = 2\left(\frac{b^2}{8a}(x_3 - x_2) - \kappa\right) \tag{140}$$

The second is $v_1 = z_1 - z_2$, and we observe

$$v_1' H^{\mathcal{J}} v_1 = 2\left(\kappa - \frac{b^2}{8a}(x_3 - x_2)\right) = -u' H^{\mathcal{J}} u \tag{141}$$

We have therefore shown the desired result but for the case in which $u' H^{\mathcal{J}} u = v_1' H^{\mathcal{J}} v_1 = 0$. Here, $x_3 - x_2 = \frac{8\tau}{\beta^2} = \frac{\Delta}{2}$. But this is precisely the case of the uniform grid.

Profits and Costs. We next derive the principal's profit and optimal tariff evaluated at the candidate uniform-grid solution:

Lemma 10. *The value to the monopolist of a K-item contract supported on a uniform grid can be written as $V(K) = \hat{\Pi}(K) - \hat{\Gamma}(K)$ where*

$$\begin{aligned} \hat{\Pi}(K) &= \frac{b - c + 2a}{4a}(2a - c) + \frac{b^2}{48a}\frac{(2K-3)(2K-1)}{(K-1)^2} \\ \hat{\Gamma}(K) &= \frac{\kappa}{2}\frac{K-2}{K-1} \end{aligned} \tag{142}$$

Moreover, the optimal allocation is supported by the tariff

$$T^*(x_k) = \frac{1}{2}\frac{k-1}{K^*-1}\left(b + c - \frac{b}{2}\frac{k-1}{K^*-1}\right) \tag{143}$$

Proof. Using the representation in Equation 22, we write

$$
\begin{aligned}
\hat{\Pi}(K) &= \sum_{k=1}^{K} \int_{\hat{\theta}_k}^{\hat{\theta}_{k+1}} \left((2a\theta + b - c)x_k - b\frac{x_k^2}{2} \right) \mathrm{d}\theta \\
&= \sum_{k=1}^{K} \left[a x_k \theta^2 - x_k \left(c - b + \frac{b}{2} x_k \right) \theta \right]_{\hat{\theta}_k}^{\hat{\theta}_{k+1}} \\
&= \sum_{k=1}^{K} \left(a x_k (\hat{\theta}_{k+1} - \hat{\theta}_k)(\hat{\theta}_{k+1} + \hat{\theta}_k) - x_k \left(c - b + \frac{b}{2} x_k \right) (\hat{\theta}_{k+1} - \hat{\theta}_k) \right) \\
&= \frac{b}{2\tau(K-1)} \sum_{k=2}^{K-1} \left(a x_k (\hat{\theta}_{k+1} + \hat{\theta}_k) - x_k \left(c - b + \frac{b}{2} x_k \right) \right) + (1 - \hat{\theta}_K) \left(a(1 + \hat{\theta}_K) + \frac{b}{2} - c \right)
\end{aligned}
\tag{144}
$$

where, in the fourth equality, we use that $\hat{\theta}_{k+1} - \hat{\theta}_k = \frac{a}{2b(K-1)}$ for $k < K$ and that $\hat{\theta}_{K+1} = 1$ and $x_K = 1$. We simplify the summation term as

$$
\begin{aligned}
&\sum_{k=2}^{K-1} \left(a x_k (\hat{\theta}_{k+1} + \hat{\theta}_k) - x_k \left(a + c - b + \frac{b}{2} x_k \right) \right) \\
&= \sum_{k=2}^{K-1} \left(a x_k \left(\frac{c-b}{a} + \frac{b}{a} x_k \right) - x_k \left(c - b + \frac{b}{2} x_k \right) \right) \\
&= \frac{b}{2} \sum_{k=2}^{K-1} x_k^2 \\
&= \frac{b}{2} \sum_{k=2}^{K-1} \left(\frac{k-1}{K-1} \right)^2 = \frac{b}{12(K-1)} (K-2)(2K-3)
\end{aligned}
\tag{145}
$$

where in the second line we use that $\hat{\theta}_k + \hat{\theta}_{k+1} = 1 - \frac{b-c}{a} + \frac{\beta}{a} x_k$. To simplify the second term, we observe that

$$
\begin{aligned}
\hat{\theta}_K &= \left(\frac{c-b}{2a} \right) + \frac{b}{4a} \left(1 + \frac{K-2}{K-1} \right) \\
1 - \hat{\theta}_K &= \frac{2(b-c)(K-1) - b(2K-3)}{4a(K-1)}
\end{aligned}
\tag{146}
$$

and therefore

$$
(1 - \hat{\theta}_K) \left(a(\hat{\theta}_K + 1) + \frac{b}{2} - c \right) = \frac{b - c + 2a}{4a} (2a - c) + \frac{b^2}{16a(K-1)^2} (2K - 3) \tag{147}
$$

Putting this together, we write

$$\begin{aligned}\hat{\Pi}(K) &= \frac{b^2}{24a(K-1)^2}(K-2)(2K-3) + \frac{b-c+2a}{4a}(2a-c) + \frac{b^2}{16a(K-1)^2}(2K-3) \\ &= \frac{b-c+2a}{4a}(2a-c) + \frac{b^2}{48a}\frac{(2K-3)(2K-1)}{(K-1)^2}\end{aligned} \tag{148}$$

We next show the desired representation of $\hat{\Gamma}$. This follows by direct calculation:

$$\begin{aligned}\hat{\Gamma}(K) &= \kappa \sum_{k=1}^{K-1} \frac{1}{K-1}\frac{k-1}{K-1} \\ &= \frac{\kappa}{(K-1)^2}\sum_{k=1}^{K-1}(k-1) \\ &= \frac{\kappa}{2}\frac{K-2}{K-1}\end{aligned} \tag{149}$$

We finally compute the tariff. Using Equation 12, we have that

$$\begin{aligned}T^*(x_k) &= \mathbb{I}[k \geq 2] \sum_{j=2}^{k} \left[u(x_j, \hat{\theta}_j) - u(x_{j-1}, \hat{\theta}_j)\right] \\ &= \mathbb{I}[k \geq 2] \sum_{j=2}^{k} \left[(a\hat{\theta}_j + b)(x_j - x_{j-1}) + \frac{b}{2}(x_{j-1}^2 - x_j^2)\right] \\ &= \mathbb{I}[k \geq 2] \sum_{j=2}^{k} \left[(a\hat{\theta}_j + b)\frac{1}{K-1} + \frac{b}{2(K-1)^2}((j-2)^2 - (j-1)^2)\right] \\ &= \mathbb{I}[k \geq 2] \sum_{j=2}^{k} \left[(a\hat{\theta}_j + b)\frac{1}{K-1} - \frac{b}{2(K-1)^2}(2j-3)\right] \\ &= \mathbb{I}[k \geq 2] \sum_{j=2}^{k} \left[\left(a\left(\frac{c-b}{2a} + \frac{b}{4a}\frac{2j-3}{K-1}\right) + b\right)\frac{1}{K-1} - \frac{b}{2(K-1)^2}(2j-3)\right] \\ &= \mathbb{I}[k \geq 2] \sum_{j=2}^{k} \left[\frac{b+c}{2(K-1)} - \frac{b}{4(K-1)^2}(2j-3)\right] \\ &= \frac{1}{2}\frac{k-1}{K^*-1}\left(b + c - \frac{b}{2}\frac{k-1}{K^*-1}\right)\end{aligned} \tag{150}$$

where we substitute in the expressions for $\hat{\theta}_j$ and x_j, simplify at each step, and evaluate at $K = K^*$. □

To derive $\tilde{K}$, we take the first derivative of V:

$$V'(K) = \frac{b^2}{24a(K-1)^3} - \frac{\kappa}{2(K-1)^2} \tag{151}$$

We observe that $V'(K) > 0$ if and only if

$$K < \tilde{K} := \frac{b^2}{12a\kappa} + 1 \tag{152}$$

We now prove that $|K^* - \tilde{K}| < 1$. If $K^* - \tilde{K} > 1$, then we know that $V(K^*-1) > V(K^*)$ as $V' < 0$ for all $K^*-1 < K < K^*$; this contradicts optimality. Similarly, if $\tilde{K} - K^* > 1$, we know that $V(K^*+1) > V(K^*)$ as as $V' > 0$ for all $K^* < K < K^*+1$; this contradicts optimality. Recall that we needed to check if the Hessian was negative definite. This is true so long as $K^* < \bar{K}$. As $\bar{K} = \frac{4}{3}\tilde{K}$, this holds whenever $\tilde{K} \geq 3$. It remains to check when $\tilde{K} \in (2,3)$ and $K^* = 3$. Direct calculation shows that indifference between $K = 2$ and $K = 3$ occurs when $\kappa = \frac{b^2}{16a}$. At this point, $\tilde{K} = 7/3$. Thus, whenever $K^* > 2$ is strictly optimal (which is when $\kappa < \frac{b^2}{16a}$), we have that $K^* < \bar{K}$. The comparative statics follow from standard monotone comparative statics arguments, after the observations that $V_{Ka} < 0$, $V_{Kb} > 0$, and $V_{Kc} = 0$. Finally, $V(3) - V(2) = \frac{1}{4}\left(\frac{b^2}{16a} - \kappa\right)$. Thus, whenever $\kappa < \frac{b^2}{16a}$ we have that $V(3) > V(2)$, which implies that $K^* \geq 3$.

Step 3: Mapping Back to the Original Problem

Since $e = 1 - x$, we observe that the optimal contract can be supported on $e_k = 1 - x_{K-k} = \frac{k-1}{K^*-1}$ with K^* self-enforcing recommendations. We next observe that, in the original problem, the IR constraint binds for type $\vartheta = 0$, who always (for any K^*) takes action $x = 1$ or $e = 0$ and receives direct payoff $\tilde{u}(0,0) = 0$. Therefore, $\tilde{T}(e_k) = T(x_{K^*-k}) - C$ where C solves $T(1) - C = 0$, and hence $C = \frac{1}{2}\left(\frac{b}{2} + c\right)$. Therefore, we calculate

$$\begin{aligned}
\tilde{T}(e_k) &= \frac{1}{2}\left(1 - \frac{k-1}{K^*-1}\right)\left(b + c - \frac{b}{2}\left(1 - \frac{k-1}{K^*-1}\right)\right) - \frac{1}{2}\left(\frac{1}{2}b + c\right) \\
&= \frac{1}{2}\left(1 - \frac{k-1}{K^*-1}\right)\left(\frac{b}{2} + c + \frac{b}{2}\frac{k-1}{K^*-1}\right) - \frac{1}{2}\left(\frac{1}{2}b + c\right) \\
&= \frac{b}{4}\frac{k-1}{K^*-1} - \frac{1}{2}\frac{k-1}{K^*-1}\left(\frac{b}{2} + c + \frac{b}{2}\frac{k-1}{K^*-1}\right) \\
&= -\frac{1}{2}\frac{k-1}{K^*-1}\left(c + \frac{b}{2}\frac{k-1}{K^*-1}\right)
\end{aligned} \tag{153}$$

We finally translate the transfers into wages by reversing the sign of the payments: $w(e_k) = -\tilde{T}(e_k)$ for all k. That is, a positive wage corresponds to a negative transfer from the agent (worker) to the principal (firm). This completes the proof.

A.14 Proof of Corollary 1

By assumption on (a, b, c, κ), there exists a maximal neighborhood $\mathcal{B}$ of a such that, for all $a' \in \mathcal{B}$, there exists a unique optimal K^* at (a', b, c, κ). Let $\mathcal{A}$ be the set of $a' \in \mathcal{B}$ such that $K^*(a') = K^*(a)$, where $K^*(a')$ is the unique optimal K^* at parameter vector (a', b, c, κ). We observe from Lemma 10 that $V = \hat{\Pi} - \hat{\Gamma}$ is submodular in a. This implies that K^* is monotone in a and therefore that $\mathcal{A} = \mathcal{B}$. By Proposition 7, we have that $\{w^*(e_k)\}_{k=1}^{K^*}$ is invariant to a conditional on K^*. Moreover, again by Proposition 7, $\{w^*(e_k)\}_{k=1}^{K^*}$ is different from $\{w^*(e_k)\}_{k=1}^{K}$ for any $K \neq K^*$ and, as $\mathcal{B}$ is the maximal neighborhood of a such that K^* is unique, whenever $a' \notin \mathcal{B}$, there exists an optimal $K^{**} \neq K^*$.

A.15 Proof of Corollary 2

We now consider the problem of maximizing total surplus subject to the implementability constraint, or in which

$$S(x, \theta) := u(x, \theta) + \pi(x, \theta) = (a\theta + b - c)\, x - b\frac{x^2}{2} \tag{154}$$

We observe that this is the same as $J(x, \theta; \hat{a}, \hat{b}, \hat{c})$ (see Equation 30) where $\hat{a} = a/2$, $\hat{b} = b$, and $\hat{c} = c$. All arguments in the proof of Proposition 7 apply to this transformed problem. The fact that $K^{*C} \geq K^*$ follows from the result in Proposition 7 that K^* decreases in c. The claim that $\tilde{K}^C = 2\tilde{K} - 1$ follows directly from the expression of $\tilde{K}$ from Proposition 7.

B Additional Technical Results

B.1 The Self-Enforcing Recommendations Representation of C

Lemma 11. *A contractibility correspondence C is regular if and only if there exist two closed sets $\underline{D} \subseteq X$ and $\overline{D} \subseteq X$ such that $0 \in \underline{D}$, $0, \overline{x} \in \overline{D}$ and for all $y \in X$:*

$$C(y) = \left[\max_{z \leq y : z \in \underline{D}} z, \min_{z \geq y : z \in \overline{D}} z \right] \tag{155}$$

Moreover, given C, $\underline{D}$ and $\overline{D}$ are unique and given by $\underline{D} = \cup_{y \in X} \min C(y) = \underline{\delta}(X)$ and $\overline{D} = \cup_{y \in X} \max C(y) = \overline{\delta}(X)$.

Proof. First, recall from Proposition 1 that C is regular if and only if there exists two functions $\underline{\delta}$ and $\overline{\delta}$ with the stated properties such that $C(y) = [\underline{\delta}(y), \overline{\delta}(y)]$. Here, we prove this result by establishing the equivalence of this representation and $(\underline{D}, \overline{D})$ representations of C.

From $(\underline{\delta}, \overline{\delta})$ to $(\underline{D}, \overline{D})$. We start with an ancillary lemma.

Lemma 12 (Fixed Point Lemma)**.** *Consider two functions $\underline{\delta}(x)$ and $\overline{\delta}(x)$ as in Proposition 1. Then for all $\underline{z} \in \underline{\delta}(X)$ and $\overline{z} \in \overline{\delta}(X)$, it holds $\underline{\delta}(\underline{z}) = \underline{z}$ and $\overline{\delta}(\overline{z}) = \overline{z}$.*

Proof. Let $\underline{z} = \underline{\delta}(x)$ for some $x \in X$. We have that $\underline{z} \in [\underline{\delta}(x), x]$. If $\underline{z} = x$, then we have that $\underline{\delta}(\underline{z}) = \underline{\delta}(x) = \underline{z}$. If $\underline{z} < x$, given property (ii) in Proposition 1, we must have $\underline{\delta}(\underline{z}) = \underline{\delta}(x) = \underline{z}$. The proof for $\overline{z} \in \overline{\delta}(X)$ is symmetric, using property (iii) in Proposition 1. □

Let $\underline{\delta}$ and $\overline{\delta}$ be as in (2) and define $\underline{D} = \underline{\delta}(X)$ and $\overline{D} = \overline{\delta}(X)$. First, observe that

$$\max_{z \leq x : z \in \underline{D}} z = \max_{z \leq x : z \in \underline{\delta}(X)} z \geq \underline{\delta}(x) \tag{156}$$

by construction. Let $\underline{z} = \max_{z \leq x : z \in \underline{D}} z$ and assume by contradiction that $\underline{z} > \underline{\delta}(x)$. If $\underline{z} = x$, then $x \in \underline{\delta}(X)$ and by Lemma 12 we have that $x = \underline{\delta}(x) < \underline{z}$, yielding a contradiction. If instead $\underline{z} < x$, then by Lemma 12 and the property (ii) of $\underline{\delta}$, we have $\underline{z} = \underline{\delta}(\underline{z}) = \underline{\delta}(x)$, yileding a contradiction. With this, we conclude that $\underline{z} = \underline{\delta}(x)$. With symmetric steps, we can show that $\min_{z \geq x : z \in \overline{D}} z = \overline{\delta}(x)$. Next, observe that necessarily we have $\underline{\delta}(0) = 0$, $\overline{\delta}(\overline{x}) = \overline{x}$, and $\overline{\delta}(0) = 0$ proving that $0 \in \underline{D}$ and $0, \overline{x} \in \overline{D}$. Finally, we need to show that $\underline{D}$ and $\overline{D}$ are closed. Take a sequence $z_n \in \underline{D}$ such that $z_n \to z$. Given that X is closed, we have that $z \in X$ and therefore $\underline{\delta}(z) \leq z$. Given that every z_n is in $\underline{D}$, Lemma 12 implies that $\underline{\delta}(z_n) = z_n$ for all n. Given that $\underline{\delta}$ is upper semi-continuous, it follows that

$$z = \lim_{n\to\infty} z_n = \lim_{n\to\infty} \underline{\delta}(z_n) \leq \underline{\delta}(z)$$

which implies that $z = \underline{\delta}(z)$ (as $z \geq \underline{\delta}(z)$) and therefore that $z \in \underline{D}$. This shows that $\underline{D}$ is closed. A symmetric argument shows that $\overline{D}$ is closed.

From $(\underline{D}, \overline{D})$ to $(\underline{\delta}, \overline{\delta})$. Fix $\underline{D}$ and $\overline{D}$ and define, as per Equation 155:

$$\underline{\delta}(x) = \max_{z \leq y : z \in \underline{D}} z\,, \quad \overline{\delta}(x) = \min_{z \geq y : z \in \overline{D}} z \tag{157}$$

It is immediate to see that both these functions are monotone increasing, such that $\underline{\delta}(x) \leq x \leq \overline{\delta}(x)$, and respectively upper semi-continuous and lower semi-continuous by Lemma 17.30 in Aliprantis and Border (2006). To see this, observe that the correspondences $x \rightrightarrows \{z \in \underline{D} : z \leq x\}$ and $x \rightrightarrows \left\{z \in \overline{D} : z \geq x\right\}$ are both upper hemicontinuous. Next, assume that $y \in [\underline{\delta}(x), x)$ and let $z = \underline{\delta}(x)$. We have $\underline{\delta}(y) \leq z$ by monotonicity. Moreover, by assumption $z \leq y$ and $z \in \underline{D}$, so that $z \leq \underline{\delta}(y)$ by definition. We then must have $z = \underline{\delta}(y)$. Symmetrically, assume that $y \in (x, \overline{\delta}(x)]$ and let $z = \overline{\delta}(x)$. We have $\overline{\delta}(y) \leq z$ by monotonicity. Moreover, by assumption $z \geq y$ and $z \in \overline{D}$, so that $z \geq \overline{\delta}(y)$ by definition. We then must have $z = \overline{\delta}(y)$. Finally, as $0 \in \overline{D}$, we have that $\overline{\delta}(0) = 0$. □

B.2 The Topology of Sets of Self-Enforcing Recommendations

Let $\underline{\Delta} \times \overline{\Delta}$ denote the space of pairs of functions $(\underline{\delta}, \overline{\delta})$ that have all the properties in Proposition 1. Recall that we endow this set with the relative topology induced by the product L_1 topology over pairs of integrable functions. Also, recall that $\underline{\mathcal{D}}$ denotes the collection of closed subsets of X that contain 0 and $\overline{\mathcal{D}}$ denotes the collection of closed subsets of X that contain 0 and $\overline{x}$. Let $\underline{\mathcal{D}} \times \overline{\mathcal{D}}$ denote the set of pairs of self-enforcing recommendations sets, $(\underline{D}, \overline{D})$. Recall that we have endowed this space with the product topology induced by the Hausdorff topology on each collection of sets.

Lemma 13. *The set $\underline{\mathcal{D}} \times \overline{\mathcal{D}}$ is compact.*

Proof. We show that each of $\underline{\mathcal{D}}$ and $\overline{\mathcal{D}}$ is compact in the Hausdorff topology. Specifically, we explicitly establish the compactness of $\overline{\mathcal{D}}$ and observe that an entirely symmetric argument applies to establish the compactness of $\underline{\mathcal{D}}$. With this, the compactness of the product space $\underline{\mathcal{D}} \times \overline{\mathcal{D}}$ follows from Tychonoff's theorem.

Observe that $\overline{\mathcal{D}} \subset \mathfrak{F}$, where $\mathfrak{F}$ is the collection of nonempty closed sets of X. Theorem 3.85 in Aliprantis and Border (2006) establishes that $\mathfrak{F}$ is compact in the Hausdorff topology. Fix a sequence $\left\{\overline{D}_n\right\}_{n\in\mathbb{N}} \subseteq \overline{\mathcal{D}}$. By the compactness of $\mathfrak{F}$, this sequence must have a subsequence $\overline{D}_{n_k}$ converging to some $\overline{D} \in \mathfrak{F}$ in the Hausdorff topology. Furthermore, we have that $0, \overline{x} \in \overline{D}_{n_k}$ for all k. Hence, by Hausdorff convergence of $\left\{\overline{D}_n\right\}_{n\in\mathbb{N}}$, we have that $0, \overline{x} \in \overline{D}$. Since $\overline{D}$ is a closed subset of X that contains 0 and $\overline{x}$, we have $\overline{D} \in \overline{\mathcal{D}}$. Finally, because the initial sequence was arbitrarily chosen, this implies that $\overline{\mathcal{D}}$ is compact. □

Next, recall that Proposition 1 and Lemma 11 imply that $\underline{\Delta}\times\overline{\Delta}$ and $\underline{\mathcal{D}}\times\overline{\mathcal{D}}$ are isomorphic via the maps:

$$(\underline{\delta}, \overline{\delta}) \mapsto (\underline{D}_{\underline{\delta}}, \overline{D}_{\overline{\delta}}) := (\underline{\delta}(X), \overline{\delta}(X)) \tag{158}$$

and, for all $x \in X$,

$$(\underline{D}, \overline{D}) \mapsto (\underline{\delta}_{\underline{D}}(x), \overline{\delta}_{\overline{D}}(x)) := \left(\max\left\{z \in \underline{D} : z \leq x\right\}, \min\left\{z \in \overline{D} : z \geq x\right\}\right) \tag{159}$$

We next prove that Hausdorff's convergence of sets of self-enforcing recommendations implies L_1-convergence of the corresponding envelope functions to the envelope functions induced by the limit sets.

Lemma 14. *Fix two sequences $\left\{(\underline{D}_n, \overline{D}_n)\right\}_{n\in\mathbb{N}} \subseteq \underline{\mathcal{D}} \times \overline{\mathcal{D}}$ and $\left\{(\underline{\delta}_n, \overline{\delta}_n)\right\}_{n\in\mathbb{N}} \subseteq \underline{\Delta} \times \overline{\Delta}$ such that $(\underline{\delta}_n, \overline{\delta}_n) = (\underline{\delta}_{\underline{D}_n}, \overline{\delta}_{\overline{D}_n})$ for all $n \in \mathbb{N}$. If $(\underline{D}_n, \overline{D}_n) \to (\underline{D}, \overline{D})$, then $(\underline{\delta}_n, \overline{\delta}_n) \to (\underline{\delta}_{\underline{D}}, \overline{\delta}_{\overline{D}})$.*

Proof. Fix two sequences as in the statement and assume that $(\underline{D}_n, \overline{D}_n) \to (\underline{D}, \overline{D})$ in the product Hausdorff topology. We only show that $\overline{\delta}_n \to \overline{\delta}_{\overline{D}}$ in L_1, as $\underline{\delta}_n \to \underline{\delta}_{\underline{D}}$ follows from an entirely symmetric argument. Together these two facts imply convergence in the L_1 product topology.

For notational simplicity, denote $\overline{\delta}_{\overline{D}} = \overline{\delta}$ and let $X_{\overline{\delta}} \subseteq X$ denote the collection of points at which $\overline{\delta}$ is continuous. Because δ is non-decreasing, it follows that $X \setminus X_{\overline{\delta}}$ is at most countable. We next show that $\overline{\delta}_n(x) \to \overline{\delta}(x)$ for all $x \in X_{\overline{\delta}}$. Before showing this, we observe that the previous claim concludes the argument because

$$\lim_{n\to\infty} \int_X |\overline{\delta}_n(x) - \overline{\delta}(x)| \, \mathrm{d}x = \lim_{n\to\infty} \int_{X_{\overline{\delta}}} |\overline{\delta}_n(x) - \overline{\delta}(x)| \, \mathrm{d}x \tag{160}$$

$$= \int_{X_{\overline{\delta}}} \lim_{n\to\infty} |\overline{\delta}_n(x) - \overline{\delta}(x)| \, \mathrm{d}x = 0 \tag{161}$$

where the first equality follows from the fact that $X_{\overline{\delta}}$ has full measure, the second equality follows from the dominated convergence theorem, and the last equality follows from the claim.

Next, fix $x \in X_{\overline{\delta}}$. Clearly, if $x = \overline{x}$, then $\overline{\delta}_n(x) \to \overline{\delta}(x)$ since they are all equal to $\overline{x}$. Thus, we next always assume that $x < \overline{x}$. We split the rest of the proof of the claim into two parts depending on whether x is inside or outside $\overline{D}$.

1. Assume that $x \in X_{\overline{\delta}} \setminus \overline{D}$ and define $\overline{D}_n(x) = \overline{D}_n \cap [x, \overline{x}]$ for all n as well as $\overline{D}(x) = \overline{D} \cap [x, \overline{x}]$. We first show that $\overline{D}_n(x) \to \overline{D}(x)$ in the Hausdorff topology. Let $Li(\overline{D}_n(x))$ and $Ls(\overline{D}_n(x))$ respectively denote the topological limit inferior and the topological limit superior of the sequence $\{\overline{D}_n(x)\}_{n\in\mathbb{N}}$.[27] Next, fix $z \in \overline{D}(x)$ and observe that $z > x$ because $x \in X_{\overline{\delta}} \setminus \overline{D}$. Because $z \in \overline{D}$ and $\overline{D}_n \to \overline{D}$, it follows that there exists a sequence $z_n \in \overline{D}_n$ such that $z_n \to z$. In particular, we must have $z_n > x$ for all n large enough, yielding that $z_n \in \overline{D}_n(x)$ for all n large enough, hence that $z \in Li(\overline{D}_n(x))$. Because z was arbitrarily chosen, this implies that $\overline{D}(x) \subseteq Li(\overline{D}_n(x))$. Next, fix $z \in Ls(\overline{D}_n(x))$. By definition of Ls, there exists a sequence $\{z_k\}_{k\in\mathbb{N}}$ such that $z_k \to z$ and $z_k \in \overline{D}_{n_k}(x)$ along a subsequence parametrized by k. Because $\overline{D}_{n_k} \to \overline{D}$ in the Hausdorff topology by assumption and $z_k \in \overline{D}_{n_k}$ for all k, it follows that $z \in \overline{D}$. Similarly, because $z_k \in [x, \overline{x}]$ for all k, it follows that $z \in [x, \overline{x}]$, yielding that $z \in \overline{D}(x)$. Because z was arbitrarily chosen, this implies that $Ls(\overline{D}_n(x)) \subseteq \overline{D}(x)$ and overall that $\overline{D}(x) = Li(\overline{D}_n(x)) = Ls(\overline{D}_n(x))$. Theorem 3.93 in Aliprantis and Border (2006) then yields that $\overline{D}_n(x) \to \overline{D}(x)$ in the Hausdorff topology. Finally, because $\overline{\delta}_n(x) = \min \overline{D}_n(x)$ for all n and $\overline{\delta}(x) = \min \overline{D}(x)$, Theorem 17.31 in Aliprantis and Border (2006) implies that $\overline{\delta}_n(x) \to \overline{\delta}(x)$.

2. Assume that $x \in X_{\overline{\delta}} \cap \overline{D}$ and define $\overline{D}_n(x)$ for all n and $\overline{D}(x)$ as above. Also, let $\operatorname{int} \overline{D}$ and $\partial \overline{D}$ respectively denote the interior and the boundary points of $\overline{D}$. First, observe that $\delta(x) = x$ because $x \in \overline{D}$. We next show that, for every $\varepsilon > 0$, there exists a point $x^\varepsilon \in \overline{D}(x)$ such that $0 < |x^\varepsilon - x| \leq \varepsilon$. We split the proof of this claim into two cases:

a. If $x \in \operatorname{int} \overline{D}$, then the claim is immediately true because $\operatorname{int} \overline{D}$ is open.
b. Assume now that $x \in \partial \overline{D}$ and, by contradiction, that there exists $\varepsilon > 0$ such that for all $z \in \overline{D}(x)$ we have either $|z - x| = 0$ or $|z - x| > \varepsilon$. Because $x < \overline{x}$, there must be $z \in \overline{D}(x)$ with $|z - x| > 0$, hence for all such z we must have $|z - x| > \varepsilon$. In turn, this implies that $\overline{D} \cap [x, x + \varepsilon] = \{x\}$, and hence that $\overline{\delta}(y) > x + \varepsilon$ for all $y \in (x, x + \varepsilon)$. With this, we have that δ is discontinuous at x, yielding a contradiction.

Next, fix $\varepsilon > 0$ and $x^\varepsilon \in \overline{D}(x)$ as above. Because $x^\varepsilon \in \overline{D}$, Hausdorff convergence implies that there exists a sequence $x_n^\varepsilon \in \overline{D}_n$ such that $x_n^\varepsilon \to x^\varepsilon$. Moreover, because $x^\varepsilon > x$, for n large enough we must have $x_n^\varepsilon > x$, hence that $x_n^\varepsilon \in \overline{D}_n$. Therefore, for all n large enough, we have

$$x \leq \overline{\delta}_n(x) = \min \overline{D}_n(x) \leq x_n^\varepsilon \tag{162}$$

[27] See for example Definition 3.80 in Aliprantis and Border (2006).

Passing to the limits we have

$$x \leq \liminf_n \overline{\delta}_n(x) \leq \limsup_n \overline{\delta}_n(x) \leq x^\varepsilon \tag{163}$$

Because ε was arbitrarily chosen, by taking $\varepsilon \to 0$ we conclude that $\lim_n \overline{\delta}_n(x) = x = \overline{\delta}(x)$, as desired. □

Before stating the main result of this section, observe that each cost function $\Gamma : \underline{\Delta} \times \overline{\Delta} \to [0, \infty]$ defined over (the envelope functions of) regular contractibility correspondences induces a cost function $\Gamma_{\mathcal{D}} : \underline{\mathcal{D}} \times \overline{\mathcal{D}} \to [0, \infty]$ defined by $\Gamma_{\mathcal{D}}(\underline{D}, \overline{D}) := \Gamma(\underline{\delta}_{\underline{D}}, \overline{\delta}_{\overline{D}})$.

Proposition 8. *If a cost function Γ is lower semi-continuous (resp. continuous) in the product L_1 topology, then $\Gamma_{\mathcal{D}}$ is lower semi-continuous (resp. continuous) in the product Hausdorff topology.*

Proof. The result immediately follows from Lemma 14. □

B.3 Gateaux Differentiability and Strong Monotonicity

In this section, we define (affine) Gateaux differentiability for cost functions Γ defined over regular contractibility correspondences. Recall that we can equivalently define each Γ over the set of pairs of functions $(\underline{\delta}, \overline{\delta}) \in \underline{\Delta} \times \overline{\Delta}$ that satisfy all the properties in Proposition 1. Unfortunately, $\underline{\Delta} \times \overline{\Delta}$ is not convex, hence we first extend our cost functions to the convex hull of $\underline{\Delta} \times \overline{\Delta}$.

Let $\underline{\mathcal{Q}} \times \overline{\mathcal{Q}} \subseteq BV(X) \times BV(X)$ denote the set of pairs of monotone increasing functions $(\underline{q}, \overline{q})$ defined over X such that $\underline{q}$ is upper semi-continuous, $\overline{q}$ is lower semi-continuous, $\underline{q}(x) \leq x \leq \overline{q}(x)$ for all $x \in X$ with equality at $x = 0$, and $\overline{q}(\overline{x}) = \overline{x}$.[28]

Lemma 15. *The set $\underline{\mathcal{Q}} \times \overline{\mathcal{Q}}$ is convex and the set of its extreme points coincide with $\underline{\Delta} \times \overline{\Delta}$.*

Proof. It is immediate to see that $\underline{\mathcal{Q}} \times \overline{\mathcal{Q}}$ is convex. Moreover, one can verify that, up to a normalization, each pair $(\underline{q}, \overline{q}) \in \underline{\mathcal{Q}} \times \overline{\mathcal{Q}}$ correspond to a pair of a CDF and quantile function over X. In particular, each $\underline{q}$ is bounded in the first stochastic dominance order between the Dirac measure at 0 and the uniform distribution, while each $\overline{q}$ is bounded in the first stochastic dominance order between the uniform distribution and the Dirac measure at 1. With this, Theorem 1 in Yang and Zentefis (2024) implies that the extreme points of $\underline{\mathcal{Q}} \times \overline{\mathcal{Q}}$ are exactly the envelope functions $(\underline{\delta}, \overline{\delta})$ defined in Proposition 1. □

[28] $BV(X)$ denotes the set of real-valued functions of bounded variation over X.

We next adapt the notion of (affine) Gateaux differentiability of Cerreia-Vioglio, Maccheroni, Marinacci, Montrucchio, and Stanca (2024) to the present setting. We say that $\tilde{\Gamma}$ is an extension of Γ over $\underline{\mathcal{Q}} \times \overline{\mathcal{Q}}$ if $\tilde{\Gamma}(\underline{\delta}, \overline{\delta}) = \Gamma(\underline{\delta}, \overline{\delta})$ for all $(\underline{\delta}, \overline{\delta}) \in \underline{\Delta} \times \overline{\Delta}$.

Definition 7. *We say that Γ is (affine) Gateaux differentiable if it admits an extension $\tilde{\Gamma}$ over $\underline{\mathcal{Q}} \times \overline{\mathcal{Q}}$ such that, for every $(\underline{q}, \overline{q}) \in \underline{\mathcal{Q}} \times \overline{\mathcal{Q}}$, there exist two continuous continuous functions $\underline{\gamma}_{\underline{q},\overline{q}} : X \to \mathbb{R}$ and $\overline{\gamma}_{\underline{q},\overline{q}} : X \to \mathbb{R}$ that satisfy*

$$\lim_{t \downarrow 0} \frac{\tilde{\Gamma}((1-t)(\underline{q}, \overline{q}) + t(\underline{r}, \overline{r})) - \tilde{\Gamma}(\underline{q}, \overline{q})}{t} = \int_X \underline{\gamma}_{\underline{q},\overline{q}}(x) \, \mathrm{d}\big(\underline{r} - \underline{q}\big)(x) + \int_X \overline{\gamma}_{\underline{q},\overline{q}}(x) \, \mathrm{d}(\overline{r} - \overline{q})(x) \tag{164}$$

for all $(\underline{r}, \overline{r}) \in \underline{\mathcal{Q}} \times \overline{\mathcal{Q}}$.[29]

It is standard to show that every uncertain cost of distinguishing is Gateaux differentiable provided that $\lambda \in (-\infty, \infty)$.

Proposition 9. *If Γ is Gateaux differentiable and there exists $\varepsilon > 0$ such that for all $(\underline{q}, \overline{q}) \in \underline{\mathcal{Q}} \times \overline{\mathcal{Q}}$ and for all $x, x' \in X$ with $x \le x'$,*

$$\underline{\gamma}_{\underline{q},\overline{q}}(x') - \underline{\gamma}_{\underline{q},\overline{q}}(x) \ge \varepsilon(x' - x) \quad \textit{and} \quad \overline{\gamma}_{\underline{q},\overline{q}}(x) - \overline{\gamma}_{\underline{q},\overline{q}}(x') \ge \varepsilon(x' - x) \tag{165}$$

then Γ is strongly monotone.

Proof. Fix $(\underline{q}, \overline{q}), (\underline{q}', \overline{q}') \in \underline{\Delta} \times \overline{\Delta}$ such that $\underline{q} \le \underline{q}'$ and $\overline{q}' \le \overline{q}$, and let C and C' denote the induced regular contractibility correspondences. We want to show that

$$\Gamma(\underline{q}', \overline{q}') - \Gamma(\underline{q}, \overline{q}) \ge \varepsilon L(C' \setminus C) = \varepsilon \left(\int_X \big(\underline{q}'(x) - \underline{q}(x)\big) \, \mathrm{d}x + \int_X (\overline{q}(x) - \overline{q}'(x)) \, \mathrm{d}x \right) \tag{166}$$

By Theorem 18 of Cerreia-Vioglio, Maccheroni, Marinacci, Montrucchio, and Stanca (2024), there exists $t \in (0,1)$ such that

$$\tilde{\Gamma}(\underline{q}', \overline{q}') - \tilde{\Gamma}(\underline{q}, \overline{q}) = \int_X \underline{\gamma}_{\underline{q}_t,\overline{q}_t}(x) \, \mathrm{d}\big(\underline{q} - \underline{q}'\big)(x) + \int_X \overline{\gamma}_{\underline{q}_t,\overline{q}_t}(x) \, \mathrm{d}(\overline{q} - \overline{q}')(x) \tag{167}$$

where $(\underline{q}_t, \overline{q}_t) = t(\underline{q}, \overline{q}) + (1-t)(\underline{q}', \overline{q}')$. By applying the Riemann–Stieltjes integral version

[29]The integrals on the right-hand side of 164 are Riemann-Stieltjes integrals.

of integration by parts, the right-hand side becomes:

$$\int_X \underline{q}'(x) - \underline{q}(x)\mathrm{d}\underline{\gamma}_{\underline{q}_t,\overline{q}_t}(x) - \int_X \overline{q}(x) - \overline{q}'(x)\mathrm{d}\overline{\gamma}_{\underline{q}_t,\overline{q}_t}(x) \tag{168}$$

$$= \int_X \left(\underline{q}'(x) - \underline{q}(x)\right) \frac{\partial}{\partial x}\underline{\gamma}_{\underline{q}_t,\overline{q}_t}(x)\mathrm{d}x + \int_X \left(\overline{q}(x) - \overline{q}'(x)\right) \left(-\frac{\partial}{\partial x}\overline{\gamma}_{\underline{q}_t,\overline{q}_t}(x)\right) \mathrm{d}x \tag{169}$$

$$\geq \varepsilon \left(\int_X \left(\underline{q}'(x) - \underline{q}(x)\right) \mathrm{d}x + \int_X \left(\overline{q}(x) - \overline{q}'(x)\right) \mathrm{d}x \right) \tag{170}$$

where the first equality and second inequality follow from (ii) since it implies that $\underline{\gamma}_{\underline{q}_t,\overline{q}_t}$ is monotone increasing, $\overline{\gamma}_{\underline{q}_t,\overline{q}_t}$ is monotone decreasing, and that they are both differentiable almost everywhere with bounded derivative. □

B.4 Incomplete Information and Incomplete Contracts

In the main text, we studied contracts with incomplete information. However, as in our applications, our analysis also applies to complete-information contracts that maximize total surplus, rather than virtual surplus. To be concrete, define total surplus as $S(x, \theta) = \pi(x, \theta) + u(x, \theta)$ and assume that this is strictly supermodular in (x, θ) and strictly quasi-concave in x. The complete-information mechanism design and contractibility problems are given by:

$$\mathcal{S}(C) := \sup_{(\phi,\xi,T)\in\mathcal{I}(C)} \int_\Theta S(\phi(\theta), \theta)\mathrm{d}F(\theta) \tag{171}$$

$$\sup_{C\in\mathcal{C}} \quad \mathcal{S}(C) - \Gamma(C) \tag{172}$$

Understanding contractibility under complete information is interesting for three reasons. First, it allows us to understand how incomplete information affects incomplete contracts. This is because the principal's problem under complete information reduces to the efficient problem.[30] Second, it is directly useful for understanding the welfare effects of incomplete contracts. Third, it allows us to study settings in which the agents have the bargaining power and choose a contract to maximize their expected utility subject to the principal's participation.[31]

[30] This is because the participation constraint of each type θ must bind under complete information and so the principal extracts full surplus from each type. Although Problem 171 is defined to include the incentive compatibility constraint implied by incomplete information, strict supermodularity of S implies that the global incentive compatibility constraint would be slack.

[31] Formally, this corresponds to the constraint that the principal's expected payoff is no less than their outside option (normalized to 0): $\int_\Theta \left(\pi(\phi(\theta), \theta) + T(\xi(\theta))\right) \mathrm{d}F(\theta) \geq 0$. It is then standard to show that this participation constraint must bind at the agent's optimal contract which in turn must solve Problem Problem 171. Therefore, the extent of optimal contractibility must again solve Problem 172.

All of our results apply to this problem, where J in our earlier results must simply be substituted with S. This observation opens up the door to comparative statics results on the extent of optimal contractibility across the incomplete-information (revenue-maximization) and complete-information (efficient) cases. For example, the new bound on the optimal extent of contractibility in the efficient case is $|\overline{D}_C^*| \leq \left\lfloor 2\left(\frac{3\overline{x}\bar{S}_{xx}^2\bar{f}}{\varepsilon \underline{S}_{x\theta}} + 1\right)\right\rfloor$, where $\overline{D}_C^*$ is any complete-information set of self-enforcing recommendations, where $\bar{S}_{xx} = \max_{x,\theta} |S_{xx}(x,\theta)|$ and $\underline{S}_{x\theta} = \min_{x,\theta} S_{x\theta}(x,\theta)$. We denote this bound by B_C, in contrast to the incomplete-information bound B derived in the main analysis. Thus, changes in concavity and supermodularity induced by information rents can be seen to directly impact the difference between efficient and revenue-maximizing contractibility. In fact, our general bounds on the completeness of contracts can be explicitly compared:

Proposition 10 (Incomplete Information Begets Incomplete Contracts)**.** *If $u_{xx\theta} \geq 0$, $u_{x\theta\theta} \leq 0$, and F satisfies the monotone hazard rate property, then $B \leq B_C$.*

Proof. We first observe observe that, for all x, θ:

$$\begin{aligned} J_{xx}(x,\theta) &= u_{xx}(x,\theta) + \pi_{xx}(x,\theta) - h(\theta)u_{xx\theta}(x,\theta) \\ &= S_{xx}(x,\theta) - h(\theta)u_{xx\theta}(x,\theta) \\ &\leq S_{xx}(x,\theta) \end{aligned} \tag{173}$$

where $h(\theta) = (1 - F(\theta))/f(\theta)$ denotes the inverse hazard rate and the last inequality uses the assumption that $u_{xx\theta} \geq 0$. We next observe that, for all x, θ:

$$\begin{aligned} J_{x\theta}(x,\theta) &= u_{x\theta}(x,\theta) + \pi_{x\theta}(x,\theta) - h'(\theta)u_{x\theta\theta}(x,\theta) - h(\theta)u_{xx\theta}(x,\theta) \\ &= S_{x\theta}(x,\theta) - h'(\theta)u_{x\theta\theta}(x,\theta) - h(\theta)u_{xx\theta}(x,\theta) \\ &\geq S_{x\theta}(x,\theta) \end{aligned} \tag{174}$$

where the last line uses the assumptions that $u_{xx\theta} \leq 0$, $u_{x\theta\theta} \geq 0$, and that the hazard rate is monotone increasing (so the inverse hazard rate, $h(\theta)$, is monotone decreasing). The result $B \leq B_C$ is immediate from combining the inequalities above with the formula for the bound. □

This result provides a general set of conditions under which our bound on the completeness of a contract is greater under complete information than under incomplete information. Intuitively, under these conditions, the curvature of total surplus is greater than that of virtual surplus and the supermodularity of total surplus is lesser than that of virtual surplus. At a more basic level, these conditions imply that the gains from contracting are lesser under incomplete information than under complete information. Thus, the presence of information rents under incomplete information naturally dampens gains from trade and thereby reduces incentives for writing more complete contracts.

While not universal, the conditions of Proposition 10 are common in applied work. In particular, the monotone hazard rate property of F and the condition that $u_{xx\theta} \geq 0$ are standard assumptions in applied theoretical work on screening (Fudenberg and Tirole, 1991; Grubb, 2009). The condition that $u_{x\theta\theta} \leq 0$ nests many papers that employ the parallel demand curves assumption that $u_{x\theta\theta} = 0$. As one example, the preferences of Mussa and Rosen (1978) (Example 1) and the preferences studied in our example (Section 4.1) satisfy all three conditions.

This result generalizes the logic of our application in Section 4.4, in which we show that incomplete information does indeed lead to more incomplete contracts by exactly solving for the optimal contracts under complete and incomplete information. The intuition there is exactly the intuition here: there are lesser gains from contracting when information is incomplete and so contracts will be more incomplete.

C An Evidentiary Foundation of Contractibility

In this section we provide a foundation of regular contractibility correspondences $C : X \rightrightarrows X$ based on a model of evidence that closely follows those of Green and Laffont (1986) and Hart, Kremer, and Perry (2017). The key difference in our model of evidence relative to these papers is that evidence is generated by agents' actions rather than their types.

C.1 Evidence and Contractibility

An evidentiary correspondence $\mathcal{E} : X \rightrightarrows \Omega$ generates for every final action of the agent $x \in X$ a set of evidence $\mathcal{E}(x) \subseteq \Omega$, where $(\Omega, \geq)$ is an arbitrary totally ordered set of possible evidence. We place the following two continuity assumptions on $\mathcal{E}$. First, for every sequence $x_n \to x$ and y, if $\mathcal{E}(x_n) \subseteq \mathcal{E}(y)$ for all $n \in \mathbb{N}$, then $\mathcal{E}(x) \subseteq \mathcal{E}(y)$. Second, for every sequence $y_n \to y$ and x such that $\mathcal{E}(x) \subseteq \mathcal{E}(y)$, there exists a subsequence $y_{n_k} \to y$ and a sequence $x_k \to x$ such that $\mathcal{E}(x_k) \subseteq \mathcal{E}(y_{n_k})$ for all $k \in \mathbb{N}$.

The principal can prove that the action taken by the agent x was not consistent with being asked to take the recommended action y if there exists a piece of evidence generated by the agent's actions $\omega \in \mathcal{E}(x)$ that could not have been generated by following the recommended action $\omega \notin \mathcal{E}(y)$. A court can impose an arbitrarily large financial penalty if the principal can prove that the agent did not act in accordance with the contract. However, the agent is innocent until proven guilty. Internalizing this, the agent would only ever take actions x that cannot be proven to be inconsistent with the recommendation y. That is, the agent would only consider taking actions x such that $\mathcal{E}(x) \subseteq \mathcal{E}(y)$. Moreover, any such action cannot be proved to be different from y, making these actions safe for the agent. This set of safe actions is given by:

$$C_{\mathcal{E}}(y) = \{x \in X : \mathcal{E}(x) \subseteq \mathcal{E}(y)\} \tag{175}$$

We call $C_{\mathcal{E}} : X \rightrightarrows X$ the contractibility correspondence induced by the evidentiary correspondence $\mathcal{E}$. We observe that our continuity assumptions on $\mathcal{E}$ immediately imply that $C_{\mathcal{E}}$ is closed-valued and lower-hemicontinuous.

Two natural conditions on the evidentiary correspondence yield regular contractibility correspondences (and vice versa):

Proposition 11. *A contractibility correspondence $C : X \rightrightarrows X$ is regular if and only if it is induced by an evidentiary correspondence $\mathcal{E} : X \rightrightarrows \Omega$ with the following properties:*

1. *Definitive evidence of exclusion: for all $x \in X$, if $\mathcal{E}(x) \subseteq \mathcal{E}(0)$, then $x = 0$.*
2. *Evidentiary monotonicity: for all $x, x' \in X$, if $x' \geq x$, then $\mathcal{E}(x') \geq_{SSO} \mathcal{E}(x)$*

Proof. **(If)** By Equation 175, it is immediate that any contractibility correspondence C that is induced by some evidentiary correspondence $\mathcal{E}$ is transitive and reflexive. Observe that definitive evidence of exclusion implies that $C(0) = \{0\}$, yielding excludability of C. It remains only to show monotonicity of C. Fix $y' \geq y$, $x \in C(y)$, and $x' \in C(y')$. If $x' \geq x$, then $\max\{x, x'\} = x' \in C(y')$ and $\min\{x, x'\} = x \in C(y)$. Suppose now that $x' < x$. We now show that $\mathcal{E}(x) \subseteq \mathcal{E}(y')$ and $\mathcal{E}(x') \subseteq \mathcal{E}(y)$, which yields monotonicity of C by the fact that these claims imply that $\max\{x, x'\} = x \in C(y')$ and $\min\{x, x'\} = x' \in C(y)$. By definition we have that $\mathcal{E}(x) \subseteq \mathcal{E}(y)$, $\mathcal{E}(x') \subseteq \mathcal{E}(y')$, $\mathcal{E}(y') \geq_{SSO} \mathcal{E}(y)$, and $\mathcal{E}(x) \geq_{SSO} \mathcal{E}(x')$. Fix $\omega \in \mathcal{E}(x)$ and $\omega' \in \mathcal{E}(x')$. If $\omega \leq \omega'$, as $\mathcal{E}(x) \geq_{SSO} \mathcal{E}(x')$, we have that $\omega = \min\{\omega, \omega'\} \in \mathcal{E}(x')$. As $\mathcal{E}(x') \subseteq \mathcal{E}(y')$, this implies that $\omega \in \mathcal{E}(y')$. If $\omega > \omega'$, as $\omega \in \mathcal{E}(x) \subseteq \mathcal{E}(y)$ and $\omega' \in \mathcal{E}(x') \subseteq \mathcal{E}(y')$, we know that $\omega = \max\{\omega, \omega'\} \in \mathcal{E}(y')$ by the fact that $\mathcal{E}(y') \geq_{SSO} \mathcal{E}(y)$. These steps imply that $\mathcal{E}(x) \subseteq \mathcal{E}(y')$. Now fix $\omega \in \mathcal{E}(x')$ and $\omega' \in \mathcal{E}(x)$. If $\omega > \omega'$, we have that $\omega = \max\{\omega, \omega'\} \in \mathcal{E}(x)$ as $\mathcal{E}(x) \geq_{SSO} \mathcal{E}(x')$. As $\mathcal{E}(x) \subseteq \mathcal{E}(y)$, this implies that $\omega \in \mathcal{E}(y)$. If $\omega \leq \omega'$, as $\omega \in \mathcal{E}(x') \subseteq \mathcal{E}(y')$ and $\omega' \in \mathcal{E}(x) \subset \mathcal{E}(y)$, we have that $\omega = \max\{\omega, \omega'\} \in \mathcal{E}(y)$ as $\mathcal{E}(y') \geq_{SSO} \mathcal{E}(y)$. These steps establish that $\mathcal{E}(x') \subseteq \mathcal{E}(y)$.

(Only If) Fix $C \in \mathcal{C}$ and define $\Omega = X$ and $\mathcal{E}(y) = C(y)$ for all $y \in X$. We first show that:

$$C_{\mathcal{E}}(y) = \{x \in X : \mathcal{E}(x) \subseteq \mathcal{E}(y)\} = \{x \in X : C(x) \subseteq C(y)\} = C(y) \tag{176}$$

where the first two equalities are by definition. The final equality follows from two observations. First, fix $x \in C(y)$ and note by transitivity of C that $C(x) \subseteq C(y)$, which implies that $C(y) \subseteq \{x \in X : C(x) \subseteq C(y)\}$. Now fix $z \in \{x \in X : C(x) \subseteq C(y)\}$, which means that $C(z) \subseteq C(y)$ and we note that $z \in C(z)$ by reflexivity, implying that $\{x \in X : C(x) \subseteq C(y)\} \subseteq C(y)$. We now establish that 1. and 2. are satisfied. Consider $\mathcal{E}(x) \subseteq \mathcal{E}(0)$, by excludability we have that $\mathcal{E}(x) \subseteq C(0) = \{0\}$. By reflexivity we have that $x \in C(x)$ and therefore $x \in \mathcal{E}(x)$. This implies that $x = 0$, establishing definitive evidence of exclusion. Monotonicity follows immediately from the fact that $\mathcal{E} = C$. $\square$

Thus, so long as higher actions generate higher evidence and the principal can always prove they excluded the agent, regularity of the contractibility correspondence is ensured. We take this as a foundation for our focus on regular contractibility correspondences.

The evidentiary model considered here is one in which the principal has evidence and the agent is considered innocent until proven guilty. However, alternative legal protocols are possible. One possibility is the opposite to what is studied above: the agent is guilty until proven innocent. Moreover, the agent has the evidence and can tell the truth and nothing but the truth, but perhaps not the whole truth. In mathematical terms, they can produce a subset $\Omega_0 \subseteq \mathcal{E}(x)$ of the evidence that they generate. It is immediate that the strongest possible evidence of innocence for an agent recommended y is $\mathcal{E}(y)$. Thus, the set of safe actions under this evidentiary standard becomes the set of actions such that $\mathcal{E}(y) \subseteq \mathcal{E}(x)$. Similar monotonicity and exclusion conditions on $\mathcal{E}$ characterize regularity of the induced contractibility correspondence. Yet further, it is possible to combine both evidentiary standards: the agent can safely take an action if $\mathcal{E}(y) \subseteq \mathcal{E}(x)$ and $\mathcal{E}(x) \subseteq \mathcal{E}(y)$, *i.e.*, $\mathcal{E}(y) = \mathcal{E}(x)$. This provides a foundation for focusing on contractibility correspondences that partition the action space. We leave a further exploration of legal procedure and contractibility to future work.

C.2 Costs of Evidence and Contractibility

We have provided an evidentiary foundation for regular contractibility correspondences. It is then natural to ask if the cost functions we have considered can be justified in the same terms. We will describe an evidentiary correspondence as regular if it satisfies both conditions of Proposition 11 and we let the set of regular evidentiary correspondences be $\mathscr{E}$. We can then define a cost function on the space of regular evidentiary correspondences as $\tilde{\Gamma} : \mathscr{E} \to [0, \infty]$.

Towards defining what it means for $\tilde{\Gamma}$ to be monotone, we first define an order over evidentiary correspondences. We say that $\mathcal{E}'$ generates more refined evidence than $\mathcal{E}$, which we denote by $\mathcal{E}' \succsim \mathcal{E}$, if for all $x, y \in X$ such that $\mathcal{E}'(x) \subseteq \mathcal{E}'(y)$ we also have that $\mathcal{E}(x) \subseteq \mathcal{E}(y)$. In words, this means that $\mathcal{E}'$ generates more refined evidence than $\mathcal{E}$ if every time that x

cannot be proven by the principal to be inconsitent with y using evidence generated by $\mathcal{E}'$, the same is true if evidence were generated by $\mathcal{E}$. Observe that $\mathcal{E}' \succsim \mathcal{E}$ if and only if $C_{\mathcal{E}'} \subseteq C_{\mathcal{E}}$. Thus, this order over evidence is equivalent to our order of having more contractibility in the main analysis.

We say that $\tilde{\Gamma}$ is monotone if whenever $\mathcal{E}' \succsim \mathcal{E}$, then we have that $\tilde{\Gamma}(\mathcal{E}') \geq \tilde{\Gamma}(\mathcal{E})$. We argue that this is a natural property for a cost to possess: if an evidentiary correspondence generates more refined evidence, then it costs more.

We now show that monotonicity of $\tilde{\Gamma}$ justifies writing costs directly over contractibility correspondences. Formally, we define what it means for a cost function to be measurable in the induced contractibility correspondence as:

Definition 8 (C-measurability)**.** *$\tilde{\Gamma}$ is $C-$measurable if there exists a $\Gamma : \mathcal{C} \to [0, \infty]$ such that for all $\mathcal{E} \in \mathscr{E}$, we have that $\tilde{\Gamma}(\mathcal{E}) = \Gamma(C_{\mathcal{E}})$.*

If $\tilde{\Gamma}$ is $C-$measurable, we call the corresponding Γ the induced cost. We can now state the following result:

Lemma 16. *If $\tilde{\Gamma}$ is monotone, then it is $C-$measurable.*

Proof. Fix an arbitrary pair of evidentiary correspondences $\mathcal{E}, \mathcal{E}' \in \mathscr{E}$ such that $C_{\mathcal{E}} = C_{\mathcal{E}'}$. We have that $C_{\mathcal{E}} \subseteq C_{\mathcal{E}'}$, which implies that $\mathcal{E} \succsim \mathcal{E}'$. By monotonicity of $\tilde{\Gamma}$ we have that $\tilde{\Gamma}(\mathcal{E}) \geq \tilde{\Gamma}(\mathcal{E}')$. By the reverse argument, we have that $\tilde{\Gamma}(\mathcal{E}') \geq \tilde{\Gamma}(\mathcal{E})$. Hence, we have that $\tilde{\Gamma}(\mathcal{E}') = \tilde{\Gamma}(\mathcal{E})$ for all $\mathcal{E}, \mathcal{E}' \in \mathscr{E}$ that induce the same contractibility correspondence. Take $\Gamma(C_{\mathcal{E}}) = \tilde{\Gamma}(\mathcal{E})$. Thus, $\tilde{\Gamma}$ is $C-$measurable with induced Γ. □

This result implies that whenever costs are monotone in the natural sense over underlying evidentiary correspondences, it is without loss of optimality to directly write the cost in the space of contractibility correspondences. This justifies the approach of so doing that we adopt in the main analysis.

We now define the relevant notion of strong monotonicity for costs over evidentiary correspondences and show that it implies strong monotonicity for the induced cost over contractibility correspondences. To do this, we first define costs of distinguishing in the space of evidentiary correspondences. For any $g \in \mathcal{G}$, we write the evidentiary cost of distinguishing as:

Definition 9 (Evidentiary Cost of Distinguishing)**.** *For any $g \in \mathcal{G}$, the evidentiary cost of distinguishing outcomes is given by:*

$$\tilde{\Gamma}^g(\mathcal{E}) = \int_{\mathcal{E}(x) \cap \mathcal{E}(y)^c \neq \emptyset} g(x, y) \, \mathrm{d}x \, \mathrm{d}y \tag{177}$$

This cost says that the principal incurs a cost of $g(x, y)$ whenever taking action $x \in X$ generates a piece of evidence $\omega \in \mathcal{E}(x)$ such that that evidence could not have been generated by following the recommendation $y \in X$ (*i.e.*, $\omega \notin \mathcal{E}(y)$). That is, if it is possible to prove x inconsistent with y, then the principal incurs the cost $g(x, y) > 0$. It is immediate to observe that an evidenitary cost of distinguishing is equivalent to a cost of distinguishing defined over contractibility with the same g:

$$\tilde{\Gamma}^g(\mathcal{E}) = \Gamma^g(C_{\mathcal{E}}) \tag{178}$$

Thus, costs of distinguishing have a natural evidentiary foundation.

As in the main analysis, we call an evidentiary cost of distinguishing linear if $g(x, y) = \kappa > 0$ and we write this cost function as $\tilde{\Gamma}^\kappa$. We can now define evidentiary strong monotonicity:

Definition 10 (Evidentiary Strong Monotonicity)**.** *An evidentiary cost function $\tilde{\Gamma}$ is strongly monotone if there exists $\varepsilon > 0$ such that, for all $\mathcal{E}, \mathcal{E}' \in \mathscr{E}$ such that $\mathcal{E}' \succsim \mathcal{E}$:*

$$\tilde{\Gamma}(\mathcal{E}') - \tilde{\Gamma}(\mathcal{E}) \geq \varepsilon \left(\tilde{\Gamma}^1(\mathcal{E}') - \tilde{\Gamma}^1(\mathcal{E}) \right) \tag{179}$$

With this, we show that evidentiary strong monotonicity implies strong monotonicity:

Proposition 12. *If $\tilde{\Gamma}$ is strongly monotone with constant $\varepsilon > 0$, then the induced Γ is strongly monotone with the same constant ε.*

Proof. We first observe that strong monotonicity of $\tilde{\Gamma}$ implies monotonicity of $\tilde{\Gamma}$. Thus, by Lemma 16, we have that $\tilde{\Gamma}$ is $C-$measurable and therefore has an induced Γ. Now fix an arbitrary pair $\mathcal{E}, \mathcal{E}' \in \mathscr{E}$ such that $\mathcal{E}' \succsim \mathcal{E}$, we have that:

$$\Gamma(C_{\mathcal{E}'}) - \Gamma(C_{\mathcal{E}}) = \tilde{\Gamma}(\mathcal{E}') - \tilde{\Gamma}(\mathcal{E}) \geq \varepsilon \left(\tilde{\Gamma}^1(\mathcal{E}') - \tilde{\Gamma}^1(\mathcal{E}) \right) = \varepsilon \left(\Gamma^1(C_{\mathcal{E}'}) - \Gamma^1(C_{\mathcal{E}}) \right) \tag{180}$$

where the first equality is by $C-$measurability, the first inequality is by strong monotonicity of $\tilde{\Gamma}$ and the final equality is by equivalence of costs of distinguishing (Equation 178). Thus, as $\mathcal{E}' \succsim \mathcal{E} \equiv C_{\mathcal{E}'} \subseteq C_{\mathcal{E}}$, we have established strong monotonicity of Γ. $\square$

Summary. We have shown that: (i) monotone costs of evidence justify writing costs directly over contractibility, (ii) strongly monotone costs of evidence yield strongly monotone costs of contractibility, and (iii) costs of distinguishing defined over evidence are equivalent to costs of distinguishing defined over contractibility. We therefore argue that the assumptions we place on costs and the main class of costs that we consider in the main analysis have an evidentiary foundation.

D Beyond Strongly Monotone Costs

In this Appendix, we discuss the boundaries of the coarse contracting prediction under alternative costs that fall outside the strongly monotone class. We show that: (i) some costs motivated by writing clauses deliver coarse contracts while some do not, (ii) costs motivated solely by enforcing contracts *ex post* do not deliver coarse contracts, and (iii) menu costs do not necessarily deliver coarse contracts.

D.1 Clause-Based Costs

One natural source for costly contractibility is a fixed cost for enumerating each relevant outcome. We say a cost is *clause-based* if it depends only on the cardinality of C, which can be interpreted as the number of *clauses* in the contract. These costs do not satisfy strong monotonicity, because they are insensitive to the structure of contractibility. Nevertheless, it is possible to recover the spirit of strong monotonicity and derive a sufficient condition for optimally coarse contracts in this class. This will highlight that the prediction of incompleteness is sensitive to the parametric structure of clause-based costs: while coarseness is guaranteed for any cost of distinguishing, not all clause-based costs will deliver incomplete contracts.

Definition 11 (Clause-Based Costs)**.** *A contractibility cost is clause-based if, for any* $C \in \mathcal{C}$*, we can write* $\Gamma(C) = \hat{\Gamma}(|C(X)|)$*, where* $\hat{\Gamma} : \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\} \to [0, \infty]$ *is a lower semi-continuous and strictly increasing function with the normalization that* $\hat{\Gamma}(2) = 0$*.*

For such clause-based costs, we will discipline the rate at which marginal costs of adding a clause decline to zero with the following definition:

Definition 12 (Clause Strong Monotonicity)**.** *We say that a clause-based* Γ*, induced by* $\hat{\Gamma}$*, is* $\beta-$*clause strongly monotone for* $\beta \in \mathbb{R}$ *if there exists* $\varepsilon > 0$ *such that:*

$$\liminf_{K \to \infty} \, (\hat{\Gamma}(K+1) - \hat{\Gamma}(K))K^{\beta} \geq \varepsilon \tag{181}$$

We illustrate clause-based costs and $\beta-$clause strong monotonicity in the following three illustrative examples:

Example 7 (Linear Costs)**.** Consider the linear cost $\hat{\Gamma}(K) = K - 2$, studied by Dye (1985) and Battigalli and Maggi (2002) in their analysis of optimally incomplete contracts. This cost is $\beta-$clause strongly monotone if and only if $\beta \geq 0$. △

Example 8 (Decreasing Marginal Costs)**.** Consider the following cost with decreasing marginal costs of additional clauses $\hat{\Gamma}(K) = \frac{1}{2} - \frac{1}{K}$, which is bounded and converges to $\frac{1}{2}$ as the number of clauses become infinite. This cost is $\beta-$clause strongly monotone if and only if $\beta \geq 1$. △

Example 5 (continuing from p. 20)**.** Consider a cost with increments that are some power of the number of clauses written so far, *i.e.,* $\hat{\Gamma}(K) - \hat{\Gamma}(K-1) = (K-2)^{\alpha}$ for some $\alpha \in \mathbb{R}$, which yields a cost $\hat{\Gamma}(K) = \sum_{k=1}^{K-2} k^{-\alpha}$. This cost is $\beta-$clause strongly monotone if and only if $\beta \geq \alpha$. △

It is obvious that any unbounded clause-based cost, such as the linear cost, implies a coarse contract. It is less obvious when coarseness will be obtained for bounded clause-based costs, such as $\hat{\Gamma}(K) = \frac{1}{2} - \frac{1}{K}$ as marginal costs converge to zero as contractibility becomes perfect. The next proposition ties the optimality of coarse contracts to $\beta-$clause strong monotonicity.

Proposition 13. *If* Γ *is clause-based and* $\beta-$*clause strongly monotone for some* $\beta < 3$*, then every optimal set of self-enforcing recommendations is finite with* $|\overline{D}^*| \leq 2 + \left\lfloor \left(\frac{6\overline{x}\bar{J}_{xx}^2\bar{f}}{\varepsilon \underline{J}_{x\theta}}\right)^{\frac{1}{3-\beta}} \right\rfloor$.

Proof. We first prove that $\overline{D}^*$ is finite. We first rule out the case in which the cardinality of $\overline{D}$ is infinite but $\overline{D} \neq X$, or contractibility is not perfect. Under clause-based costs, $\Gamma(\overline{D}) = \Gamma(X)$, or there is no increase in cost to consider perfect contractibility. However, $\mathcal{J}(X) \geq \mathcal{J}(\overline{D})$. Therefore, there must also be a solution with perfect contractibility. It will therefore suffice to show that perfect contractibility cannot be optimal.

To do this, we show that there is a strict payoff improvement from replacing perfect contractibility with a uniform grid of K points, evenly spaced with width $\overline{x}/K$. Recall that ϕ^P denotes the assignment under perfect contractibility, let ϕ_K^* denote the assignment under the grid, and let $G_K = \{\overline{x}i/K\}_{i=1}^K \in \overline{\mathcal{D}}$ denote the grid. To derive the benefits of this contractibility correspondence, we apply a close variant of Lemma 2. Using the bound derived in the proof of that result for $|J(\phi^P(\theta),\theta) - J(x,\theta)|$ for any x, we derive

$$\begin{aligned} \mathcal{J}(X) - \mathcal{J}(G_K) &= \int_0^1 (J(\phi^P(\theta),\theta) - J(\phi_n^*(\theta),\theta))\, \mathrm{d}F(\theta) \\ &\leq \int_0^1 \frac{1}{2K^2} \bar{J}_{xx}\, \mathrm{d}F(\theta) = \frac{1}{2K^2} \bar{J}_{xx} \end{aligned} \tag{182}$$

We next observe that, if costs are clause strongly monotone, for sufficiently large n

$$\Gamma(X) - \Gamma(G_K) \geq \sum_{j=K}^{\infty} j^{-\beta} \varepsilon \tag{183}$$

If $\beta \leq 1$, then $\Gamma(X) - \Gamma(G_K) = \infty$ and it is clearly preferred to set G_K. If $\beta > 1$, then we note that

$$\Gamma(X) - \Gamma(G_K) \geq \varepsilon \sum_{j=K}^{\infty} j^{-\beta} \geq \varepsilon \int_K^{\infty} s^{-\beta} \, \mathrm{d}s = \varepsilon \left[-\frac{1}{\beta} s^{-\beta+1} \right]_K^{\infty} = \frac{\varepsilon}{\beta} K^{-\beta+1} \tag{184}$$

where the first inequality uses the fact that $s^{-\beta}$ is a decreasing function for $s > 0$, and therefore the integral is smaller than its approximation via left end-point steps (*i.e.,* the sum). In this case, we have

$$\mathcal{J}(G_K) - \Gamma(G_K) \geq \mathcal{J}(X) - \Gamma(X) + \left(\frac{\varepsilon}{\beta} K^{-\beta+1} - \frac{1}{2} \bar{J}_{xx} K^{-2} \right) \tag{185}$$

Yielding a contradiction to optimality for $\beta < 3$:

$$K > \left(\frac{\beta}{2\varepsilon} \bar{J}_{xx} \right)^{\frac{1}{3-\beta}} \to \mathcal{J}(G_K) - (\Gamma(G_K) - \mathcal{J}(X) - \Gamma(X)) \geq 0 \tag{186}$$

We now derive the bound on the number of clauses. Our overall strategy will be to show that, if the number of clauses exceeded the claimed upper bound, then we could remove one clause and achieve a strict improvement. We first observe that, in a K clause contract, there must exist some ordered triple of points (x_{m-1}, x_m, x_{m+1}) such that $x_{m+1} - x_{m-1} < 2\overline{x}/(K-2)$. Otherwise, there would be a contradiction:

$$\begin{aligned} x_K - x_1 = \sum_{j=1}^{\lfloor K/2 \rfloor} x_{2j+1} - x_{2j-1} &\geq \lfloor K/2 \rfloor \frac{2\overline{x}}{K-2} \\ &> \left(\frac{K}{2} - 1 \right) \frac{2\overline{x}}{\frac{K}{2} - 1} > \overline{x} \end{aligned} \tag{187}$$

We first apply the third statement of Lemma 2 to bound the loss from eliminating contractibility at some point x_m:

$$\begin{aligned} \mathcal{J}(\overline{D}^*) - \mathcal{J}(\overline{D}^* \setminus \{x_m\}) &\leq 3 \frac{\bar{J}_{xx}^2 \bar{f}}{\underline{J}_{x\theta}} (x_m - x_{m-1})(x_{m+1} - x_m)(x_{m+1} - x_{m-1}) \\ &\leq \frac{3}{4} \frac{\bar{J}_{xx}^2 \bar{f}}{\underline{J}_{x\theta}} (x_{m+1} - x_{m-1})^3 \end{aligned} \tag{188}$$

where in the second inequality we use the fact that $\max_{w+y \leq z} wy = z^2/4$. Next, applying the clause strong monotonicity of $\Gamma(D) = \hat{\Gamma}(n(D))$ to a K-clause contract, we have

$$\hat{\Gamma}(K) - \hat{\Gamma}(K-1) \geq \varepsilon(K-1)^{-\beta} > \varepsilon(K-2)^{-\beta} \tag{189}$$

A sufficient condition for the principal to prefer to remove contractibility at point x_m is if the lower bound on cost reduction is larger than the upper bound on benefits loss, or

$$\varepsilon(K-2)^{-\beta} > \frac{3}{4}\frac{\bar{J}_{xx}^2 \bar{f}}{\underline{J}_{x\theta}}(x_{m+1} - x_{m-1})^3 \tag{190}$$

We now take $x_{m+1} - x_{m-1} < 2\overline{x}/(K-2)$ and re-arrange this to

$$K > 2 + \left(\frac{6\overline{x}\bar{J}_{xx}^2 \bar{f}}{\varepsilon \underline{J}_{x\theta}}\right)^{\frac{1}{3-\beta}} \tag{191}$$

Thus, if K exceeds the right hand side, then we have found a contradiction to the optimality of the clause-based contract. □

Our step of calculating the value of a coarse contract in in the proof of Proposition 13 has precedents in the literature. In particular, Wilson (1989) shows under perfect information that coarsening the domain of contractibility into uniform cells is second-order in the length of the grid. Extending these ideas, Bergemann, Yeh, and Zhang (2021) show that this remains true with private information. By contrast, our earlier arguments away from clause-based costs that must consider set-valued perturbations are without precedent to our knowledge. The third step shows that, when costs are $\beta-$clause strongly monotone for $\beta < 3$, there is a fine enough grid that beats perfect contractibility, thereby contradicting that any infinite-support contractibility is optimal. Finally, the bound follows from using a similar argument to contradict the optimality of points spaced too close together.

To illustrate this result, let us return to the example $\hat{\Gamma}(K) = \frac{1}{2} - \frac{1}{K}$. As this cost is β-clause strongly monotone for $\beta = 1 < 3$, we have that the optimal contract is necessarily coarse. Moreover, we have a bound on the number of clauses which is given by $2 + \left\lfloor \sqrt{\frac{6\overline{x}\bar{J}_{xx}^2 \bar{f}}{\underline{J}_{x\theta}}} \right\rfloor$. Thus, despite the fact that the marginal cost of additional clauses converges to zero, there is nevertheless a finite bound on the number of clauses.

When a cost function is not β-clause strongly monotone for $\beta < 3$, it is possible that an optimal contract will be complete. Thus, the issue of whether contracts are complete hinges on the returns-to-scale in contracting in the clause-based case. In the next result, we show that if Γ is not β-clause strongly monotone for $\beta < 3$, then optimal contracts can fail to be coarse:

Proposition 14. *There exist a clause-based* Γ *that is not* $\beta-$*clause strongly monotone for some* $\beta < 3$ *and* (u, π, F) *such that* $|C^*(X)| = 2^{\aleph_0}$.

Proof. Fix, as in Appendix A.13, $u(x, \theta) = (a\theta + b)x - b\frac{x^2}{2}$, $\pi(x, \theta) = -cx$, and $F(\theta) = \theta$ with $\Theta = [0, 1]$ and $X = [0, 1]$ and suppose that $b \le c \le 2a$. We now provide a Γ that is not $\beta-$clause strongly monotone for some $\beta \le 3$ under which the optimal set of self-enforcing recommendations is $\overline{D}^* = [0, 1]$. In particular, take $\hat{\Gamma}(K) = \gamma \sum_{k=1}^{K-2} k^{-\alpha}$ for $\gamma > 0$. By construction $\hat{\Gamma}$ is β-clause strongly monotone if and only if $\beta \ge \alpha$. Computing the first-order condition from Proposition 6 yields, by identical calculations to Appendix A.13 but noting that the partial derivatives of Γ_0 are all zero, that the uniform grid $x_k = \frac{k-1}{K-1}$ is the optimal choice of $\overline{D}$ for any fixed K. Thus, as in Appendix A.13, we have that the principal's total virtual surplus is given by $\hat{\Pi}$. The claim then follows if we can show that, for all $K \ge 3$ that:

$$\hat{\Pi}(K) - \hat{\Pi}(K-1) > \hat{\Gamma}(K) - \hat{\Gamma}(K-1) \tag{192}$$

as $\hat{\Pi}(2) > \hat{\Gamma}(2) = 0$. By construction, we have that $\hat{\Gamma}(K) - \hat{\Gamma}(K-1) = \gamma(K-2)^{-\alpha}$. Thus, this inequality becomes:

$$\frac{b^2}{48a\gamma}\left[\frac{(2K-3)(2K-1)}{(K-1)^2} - \frac{(2K-5)(2K-3)}{(K-2)^2}\right] > (K-2)^{-\alpha} \tag{193}$$

This is equivalent to:

$$\frac{b^2}{12a\gamma}(2K-3) > (K-2)^{2-\alpha}(K-1)^2 \tag{194}$$

We can rewrite this as:

$$\frac{b^2}{6a\gamma} + \frac{b^2}{12a\gamma}\frac{1}{K-2} > (K-2)^{1-\alpha}(K-1)^2 \tag{195}$$

A sufficient condition for this is that:

$$\frac{b^2}{6a\gamma} > (K-2)^{1-\alpha}(K-1)^2 \tag{196}$$

If $\alpha \ge 3$, the right-hand-side is a strictly decreasing function of K. Thus, the right-hand side is maximized at $K = 3$ and so $\overline{D}^* = X$ if $\frac{b^2}{24a\gamma} > 1$. Moreover, if $\alpha \ge 3$, Γ is not $\beta-$clause strongly monotone for any $\beta < 3$, completing the proof. □

We finally observe that the characterization of the optimally chosen self-enforcing recommendations in the clause-based case is the same as the characterization in Proposition 6 with a further simplification that the marginal cost term in the right of Equation 23 is zero, as there is no contractibility cost of changing the value of any x_k. Bergemann, Shen, Xu, and Yeh (2012) have previously studied this problem of optimally spacing grid points given an exogenous constraint in the setting with linear-quadratic preferences and found the same first-order condition that we have in this case. Relative to this work, we have shown how to optimally choose such points in the presence of costs and, more substantively, how many points the principal should elect to choose.

D.2 Back-End Costs of Contractibility

We have interpreted costly contractibility as something borne *ex ante*, or before the agent takes an action. As we argued above, this could capture the principal's front-end cost of describing different outcomes in a legally precise way. A different foundation for costs could instead focus on back-end costs that are borne *ex post*, or after the agent takes (or attempts to take) an action. This could capture the expected cost of detecting a deviation from the contract or litigating a deviation from the contract, more reminiscent of the classic literature studying costly verification.

To shed light on the difference between these models, we show how an *ex post* variant of our costs of distinguishing outcomes leads to optimally *complete* contracts. The reason turns out to be simple: *ex post* costs are equivalent to additional production costs for the principal, which do not by themselves induce coarseness. We use this observation to discuss the applicability of our coarse-contracts prediction to scenarios in which one might expect more costs to be borne *ex ante* vs. *ex post*.

As described in the main text, an *ex post* cost of distinguishing is:

Definition 13. *For any $g \in \mathcal{G}$ and a recommendation function $\xi : \Theta \to \mathbb{R}$, the ex post cost of distinguishing outcomes is:*

$$\Gamma^g(C, \xi) = \int_X \int_{X \setminus C(y)} g(x, y) \, \mathrm{d}x \, \mathrm{d}F_\xi(y) \tag{197}$$

where $F_\xi(y) = \mathbb{P}_F[\xi(\theta) \leq y]$.

This differs from the *ex ante* cost of distinguishing as the total cost is evaluated under the distribution of x that obtains *ex post*, which is F_ξ, rather than under the uniform measure, which is relevant when costs are borne *ex ante*.

This example hints at a fundamental difference between *ex ante* and *ex post* costs of distinguishing outcomes: *ex post* costs are linearly separable over types while *ex ante* costs are not. The only thing that ties different types together is $\overline{\delta}$, as this is common to all types. However, under any Obedient mechanism, we know that $\phi(\theta) = \overline{\delta}(\phi(\theta))$. Thus, fixing ϕ, we have pinned down $\overline{\delta}$, and the induced cost function is linearly separable over types in their final actions. Hence, it is as if *ex post* costs of distinguishing actions are a production cost. This logic yields the following result, which implies that optimal contracts are never coarse under *ex post* costs:

Proposition 15 (*Ex Post* Costs Do Not Yield Coarse Contracts)**.** *Under ex post costs of distinguishing outcomes, if $g(x,\cdot)$ is a decreasing function for all $x \in X$, then free disposal, $C(x) = [0,x]$ for all $x \in X$, is optimal.*

Proof. We start by using the change of variables formula for pushforward measures to rewrite the cost as:

$$\begin{aligned}\Gamma^g(C,\xi) &= \int_X \int_{X\setminus C(y)} g(x,y)\mathrm{d}x\mathrm{d}F_\xi(y) = \int_\Theta \int_{X\setminus C(\xi(\theta))} g(x,\xi(\theta))\,\mathrm{d}x\,\mathrm{d}F(\theta) \\ &= \int_\Theta \left[G(\underline{\delta}(\xi(\theta)),\xi(\theta)) + G(\overline{x},\xi(\theta)) - G(\overline{\delta}(\xi(\theta)),\xi(\theta))\right]\mathrm{d}F\,(\theta)\end{aligned} \tag{198}$$

Setting $\underline{\delta} = 0$ is without loss of optimality as $\overline{\delta}$ characterizes the set of implementable allocations by Lemma 9 and always weakly lowers costs by setting $G(\underline{\delta}(\xi(\theta),\xi(\theta)) = 0$. Given this, we can further simplify the cost as:

$$\Gamma^g(C,\xi) = \int_\Theta \left[G(\overline{x},\xi(\theta)) - G(\overline{\delta}(\xi(\theta)),\xi(\theta))\right]\mathrm{d}F\,(\theta) \tag{199}$$

By Obedience, we know that $\phi(\theta) = \overline{\delta}(\phi(\theta))$. Thus, we can write:

$$\Gamma^g(C,\xi) = \int_\Theta \left[G(\overline{x},\xi(\theta)) - G(\phi(\theta),\xi(\theta))\right]\mathrm{d}F\,(\theta) \tag{200}$$

Moreover, we have that:

$$G(\overline{x},\xi(\theta)) - G(\phi(\theta),\xi(\theta)) = \int_{\phi(\theta)}^{\overline{x}} g(z,\xi(\theta))\,\mathrm{d}z \geq \int_{\phi(\theta)}^{\overline{x}} g(z,\phi(\theta))\,\mathrm{d}z \tag{201}$$

as $g(x,\cdot)$ is a decreasing function for all $x \in X$ and $\phi(\theta) \geq \xi(\theta)$ by obedience. Thus, it is without loss of optimality to set $\xi = \phi$. Given this, we have by Lemma 9 that the principal's problem can be written as:

$$\max_{\overline{\delta}} \max_{\phi \text{ Monotone: } \phi(\Theta)\subseteq\overline{\delta}(X)} \int_\Theta \left[J(\phi(\theta),\theta) - (G(\overline{x},\phi(\theta)) - G(\phi(\theta),\phi(\theta)))\right]\mathrm{d}F\,(\theta) \tag{202}$$

We observe that the constraint $\phi(\Theta) \subseteq \overline{\delta}(X)$ is least restrictive when $\overline{\delta}(X) = X$ and so this is optimal. Given this, we have that $C(x) = [0,x]$ for all $x \in X$ is optimal. □

Thus, the optimal contract makes it impossible to choose an action in excess of a recommendation, $\overline{\delta}(x) = x$, but allows for the possibility of free disposal. This generates no loss in value for the principal but economizes on the costs of monitoring for disposal, which they know will never actually happen as the agent has a positive marginal value for all units of the good. That is, the contracting outcomes are *as if* contractibility is perfect.

Remark 7 (Mixing *Ex Ante* and *Ex Post* Costs of Distinguishing)**.** Realistic scenarios might be described as a combination of both *ex ante* and *ex post* costs of distiguishing. That is, a principal may both have to write a contract that precisely distinguishes actions and enforce it. We might model such scenarios by allowing the "true" cost faced by the principal to be a weighted sum of *ex ante* and *ex post* costs. For instance, in the context of the aforementioned examples, we could have:

$$\Gamma(C, \xi) = \nu\Gamma^g(C) + \Gamma^g(C, \xi) \tag{203}$$

for some $\nu \in \mathbb{R}_+$, where $\Gamma^g(C)$ is some cost of distinguishing outcomes and $\Gamma^g(C, \xi)$ is some *ex post* cost of distinguishing outcomes. Provided that $\nu > 0$, Theorem 1 holds and optimal contracts are coarse. Moreover, the bound in Theorem 1 decreases in ν. △

Thus, our theory predicts coarser contracts in scenarios in which defining outcomes *ex ante* is particularly difficult compared to scenarios in which outcomes are very well defined but merely difficult to detect, punish, or enforce. The first category might include variable quality services like hotel stays, vehicle rentals, or management consulting. What these scenarios have in common is that "success," "quality," and/or "damage" are inherently difficult to define. While there are surely issues also with enforcement, at least some meaningful fraction of costs comes from designing the contract in the first place ($\nu > 0$). The second category might include metered utilities, in which the sole difficulty is the precise measurement of *ex post* usage.

D.3 Menu Costs

Another natural source of non-production costs for the principal are *menu costs* of various forms: that is, costs of putting products up for sale rather than costs of delivering the final product *per se*. A rich class of menu costs can be described by the expanded class of costs $\Gamma(C, \xi)$. For example, our baseline costs of distinguishing actions can be re-interpreted as a type of menu cost that leads to coarse contract. Clause-based costs, which depend on the cardinality of the menu, can be interpreted as a menu cost that may or may not induce coarse contracts. In general, however, not all reasonable menu costs induce coarse contracts, as we argue in the following eample.

Example 9 (Menu Costs from Maximum Quality)**.** Consider the cost function studied by Sartori (2021), in which the indirect cost of a menu corresponds to the cost of the most expensive quality to be produced. Formally, fix a continuous and increasing baseline cost function $c : X \to \overline{\mathbb{R}}$ and define

$$\Gamma(C, \xi) = \max_{x \in \xi(\Theta)} c(x) \tag{204}$$

The interpretation of this cost function is that the monopolist invests ex-ante in a maximum level of quality x of the good and then they are able to freely garble this quality by offering any smaller level $y \leq x$. It is easy to see that Γ does not satisfy the strong monotonicity properties of Section 3, since it depends only on the largest (relevant) item on the menu. In fact, the analysis in Sartori (2021) shows that, in general, the optimal menu offered by the monopolist is not coarse and involves a continuum of differentiated qualities. $\triangle$